

EMERGENT SYMMETRIES:
A Group Theoretic Analysis
of an Exemplar of Late Modernism: the Smith House by Richard Meier

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EMERGENT SYMMETRIES:
A Group Theoretic Analysis
of an Exemplar of Late Modernism: the Smith House by Richard Meier

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‘The diverse elements of Classical Architecture are organized into coherent wholes by means of geometric systems of proportion. Precise rules of axuality, symmetry, or formal sequence govern the organization of the whole with hierarchical distribution.’

‘What Modern Architecture brings out, is the complexification of these systems by algebrization of their geometric relations.’

W. J. Mitchell, ‘The Logic of Architecture’

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SUMMARY

Formal systems in architectural design aim at the systematic description, interpretation, and evaluation of existing works of architecture as well the systematic creation of new works of architecture. A basic algorithmic structure is reviewed and various examples of such systems are presented.

The recent emphasis of architecture discourse on issues of pattern making and parametric variation reaffirms the traditional role of symmetry and extends the research in new trajectories. Some basic questions regarding the extent of fitness and value of symmetry in formal composition remain unanswered. Currently all formal analysis using group theoretical tools focus on repetitive designs that show immediately their recursive structure. It is suggested here that highly complex designs can still be described and analyzed with group theoretical manner.

This work builds upon recent methodological approaches in the field (March 1998); (Park 2000); (Economou 2001) and proposes a model that investigates whether the combination of existing group theoretical formalisms with appropriate systems of representation can indeed cast new light in analysis of such works and therefore construct a rigorous body of foundational research in formal composition in architecture design. The broader question that is opened up here is whether a complex architecture object – or part depending on the interest of the researcher, can be interpreted as a layered object whose parts are all related symmetrically; in other words whether an asymmetric shape or configuration can be understood in terms of nested arrangements of some order of symmetry.

The object of analysis has been polemically selected here to be the NY5 architecture, a set of designs that are all clearly exemplifying formal qualities of abstraction, layering, complexity, depth and so on, all appearing impenetrable to a systematic and rigorous analysis using the existing group theoretical formal methods. For example, Richard Meier's work has been presented here as a hyper-refinement of the modernist imagery that has been inspired not by machines but by other architecture that was inspired by machines and especially Le Corbusier; similarly, the group formalism that can describe Meier's architecture could constitute a hyper-refined construction that relies on specific representations and mappings that foreground internal complex relationships of the structure itself, i.e. the symmetry subgroups and super-groups of any

given spatial configuration. This analogy far as it goes has its limitations too, and the same exist for many other implicit theses herein.

The computation is entirely visual. A reassembly of the layered symmetries explains the structure of the symmetry of the house and provides an illustration of the basic thesis of this research on the foundation of a theory of emergence based on symmetry considerations. All plans of the house are represented in three different levels of abstraction moving successively away from the architectural representation to a purely diagrammatic one that foregrounds divisions of space. All representations are fed into an analysis algorithm to pick up all symmetry relationships and the parts are constructed as instances of a binary composition of a family of rectangular grids. Finally the process is reversed to fully account for the construction of the space of the house as a three dimensional layered composition.

At the end, this research points to a series of other extensions and domains. These extensions generally fall into two categories; a) on the improvement of the system itself; and b) on the interpretative capabilities it affords for the construction and evaluation of critical languages of design.

Chapter 1 Introduction

Chapter 1 provides the setting for this work, including the motivations, aspirations and contributions of the research. The section provides a brief overview of formal methods in design and positions this work within this wider milieu and particularly within other approaches that use group theoretical tools. The section concludes with an outline of the dissertation.

1.1. Prelude

Formal systems in design have been used for systematic studies in analysis and synthesis of form for a long time and with a great degree of success. There are several generous accounts of their history and logic - see, for example, (March and Stiny 1985), (Kalay 2004). Among these methods the group theoretical approach has been particularly successful (Weyl 1952). This could hardly be otherwise as long as group theory provides the mathematical language for symmetry and symmetry has been one of the cornerstones of formal composition in architectural design and in the arts in general (Shubnikov and Koptsik 1974). The recent emphasis of contemporary architecture discourse on issues of pattern making and parametric variation only reaffirms the traditional role of symmetry and structural repetition as a ubiquitous and indispensable principle of composition in architectural design. Still, even if contemporary emphasis on pattern making extends the research in new trajectories including space-packing techniques, layer stacking, periodicity and non-periodicity, and so on, some basic questions regarding the value and fitness of symmetry in formal composition remain unanswered. More specifically, it remains unclear whether an apparently complex plan can be described or interpreted in a group theoretical way or not. There is a great body of work on the description of the symmetry properties of architectural works of Palladio, Soane, Ledoux, Wright, Le Corbusier and others [see for example (March and Steadman 1971)]; all of these designs typically exemplify their apparent correspondences and the power of the method is immediately appreciated. Still, there is a great body of architecture work that the power of the method seems inadequate to explain. For example, a great number of designs and especially those of late modernity in the twentieth century cannot be easily explained with existing tools. Some first steps towards the extension of the tools of group theory to explain these designs have been taken by March (1998), Park (2000) and Economou (1999), (2001).

This work builds upon this methodological approach and proposes a model that investigates whether the combination of existing group theoretical formalisms with appropriate systems of representation can indeed cast light in the analysis of such works and therefore construct a rigorous body of foundational research in formal composition in architecture design. The broader question that is opened up here is whether a complex architecture object – or part depending on the interest of the researcher, can be interpreted as a layered object whose parts are all related symmetrically; in other words whether an asymmetric shape or configuration can be understood in terms of nested arrangements of some order of symmetry.

1.2. Method

A fascinating aspect of symmetry is that it can provide a measure regarding the formal structure of an object; it tells the number of the parts that the object consists of and the ways these parts combine. This quest for an aesthetic measure is closely related to the efforts of, say, George Birkhoff (1933) to realize aesthetic formalisms or, for that matter, of all the ancient Greek mathematicians and their work on the theory of means (Heath 1932). This formal grounding of symmetry on mathematical grounds and in specific group theory has provided an approach that has generated several applications in analysis and synthesis of objects that are composed by identical parts. Classical accounts of applications in analysis and synthesis in formal composition in the visual arts have been given by March and Steadman (1971), Shubnikov and Koptsik (1974) and more recently by Park (2000) and Economou (2001). Still, this approach does not look as powerful in the analysis of designs that do not exhibit an apparent repetition in their structure. A classic example of such designs is the NY5 architecture, a set of designs that are all clearly exemplifying formal qualities of abstraction, layering, complexity, depth and so on. The key idea that is used here is that these representations of these complex objects can be understood as layered compositions of simpler parts and that these parts can all be related through symmetry values. The basic tool from group theory that is used here is the partial order lattice that pictorially presents the symmetry structure of any spatial configuration; the number and qualities of the symmetry subgroups found in any given configuration provide the maximum number of layers that can be found in a spatial configuration; for example, in any spatial arrangement that is based on the structure of the square the maximum number of layers and spatial constructs that can be build upon those is ten because this is the number of symmetry subgroups of the square. Still, the symmetry subgroups can only provide the logical framework to compute an architectural

composition; what is critical is the representation of the designs that are going to be analyzed within this framework.

This work suggests three aspects of representation to be computed within these subgroups: the first built on abstraction, the second on weighting, the last on projection. All representations rely on successive deletions of features of architectural representation. There are three levels that are suggested here: a) the first level, the architectonic level, retains all the conventions of projection and section of architectural drawings: walls, windows, doors, stairs, parapets, encased furniture, tiling, rails all represented as arrangements of lines. The next level of abstraction, the spatial level, records only topological relationships and aspects of connectivity: walls and openings of all kinds. The third and most abstract level of abstraction, the diagrammatic level, records only divisions of space. Finer distinctions of space and notations are all recorded in these drawings with a weighting of lines to show materiality, transparency, or simply other kinds of experiential relations of spatial elements next to each other. Three types of notations of lines are used here: Solid, thin, and dotted. It is suggested that this method of representation based on three types of levels of abstraction and three types of lines provide a rich repertory of devices to be computed in partial order lattices and show essential relationships in complex architectural arrangements.

1.3. Contribution

The formal techniques and methods that are developed here can be used in a variety of ways in the analysis and synthesis of form. The formal theory is applied mathematics, in particular group theory and combinatorics. The use of group generators in the generation of symmetry groups and subgroups, the use of lattices in the partial ordering of sub-symmetries of a design, and the use of the cycle index of a permutation group of a given set are three key tools used extensively throughout this research.

The object of analysis has been polemically selected here to be the NY5 architecture, a set of designs that are all clearly exemplifying formal qualities of abstraction, layering, complexity, depth and so on. One specific case study has been selected in particular to fully illustrate the methodology of analysis, the Smith House by Richard Meier. All plans of the house are represented in three different levels of abstraction moving successively away from the architectural representation to a purely diagrammatic one that foregrounds divisions of space. All representations are fed into an analysis algorithm to pick up all symmetry relationships and the

parts are constructed as instances of a binary composition of a family of rectangular grids. Finally the process is reversed to fully account for the construction of the space of the house as a three dimensional layered composition.

1.4. Outline

The dissertation is roughly divided in three parts; the first three chapters present the problem statement of the sub-symmetry analysis and provide a literature review of the general class of methods that this problem belongs to as well a state-of-the-art account of the specific methods that have solved other aspects of this problem. The following chapter provides the hypothesis and methodology of the sub-symmetry analysis attempted here and the following chapter provides one case study to test the methodology and its value. A discussion of future research directions and a summary of the work conclude this research. More specifically, the research work here is proposed in the following parts:

Chapter 1 presents the case for the research and contextualizes its position within the current state of architecture discourse on formal methods in design.

Chapter 2 provides a literature review of formal methods in design and focuses on the history of the applications of these methods in analysis and synthesis in architectural design with an emphasis on group theoretical applications.

Chapter 3 presents the logic of the system adopted here, the group theory. All basic formal constructs that are used in the research are presented here.

Chapter 4 provides the hypothesis and methodology of this work. Currently all formal analysis using group theoretical tools focus on repetitive designs that show immediately their recursive structure. It is suggested here that highly complex designs can still be described and analyzed with group theoretical manner. The key idea is that the complexity of these designs can be seen as an aggregation of spatial layers that can all be decomposed by the subgroup relations of the symmetry of the configuration.

Chapter 5 shows the application of this methodology in analysis using Richard Meier's Smith House as its major focus. All plans of the house are decomposed and abstracted in various ways and the computation of all symmetry parts takes place in entirely visual terms. The computation is entirely visual. A reassemble of the layered symmetries explains the structure of the symmetry of

the house and provides an illustration of the basic thesis of this research on the foundation of a theory of emergence based on symmetry considerations.

Chapter 6 provides a summary of the work, an assessment of its strengths and limitations and suggests future work.

Chapter 2 Formal systems in architectural design

Formal systems in architectural design aim at the systematic description, interpretation, and evaluation of existing works of architecture as well the systematic creation of new works of architecture. A basic algorithmic structure for the foundation of formal systems is reviewed (Stiny and Gips 1978) and various examples of such systems are presented. The chapter concludes with an informal presentation of applications in formal analysis and design based on group theory.

2.1. Introduction

'What makes me tick is an aesthetic sense of order, of essential simplicity behind apparent complexity. As an artist, it is possible to create exuberant and unique objects from a small and limited set of elements and rules; as a scientist, it is a challenge to discover a simple explanation for complex behavior, a general causal structure for a series of related but unique events. In this view, science and art are both aesthetic activities: only the direction of the approach differs.' March (1972).

The desire to speculate architectural design as a form of a logical construct has a long history (see for example, (Stiny and March 1981), (Kalay 2004). Particularly interesting are the efforts in the 1960's to formalize architecture in terms of some mathematical framework when design methods in architecture were associated with operational research, economics and decision theory (see for example, (Archer 1970), (Martin 1967), (Simon 1994). Integral in such a world-making is the construct of reasoning as the process of extending a set of known facts, beliefs or observations by applying to them rules that combine the known facts in a manner that produces new facts and rules. Lionel March (1976) suggested Peirce's three modes of inference as the three possible plausible reasoning in science and in design (March 1976), (Shin 1994). Whereas the major goal of scientific endeavor is to establish general laws or theory, the prime objective of designing is to realize a particular case or design. Both require deduction for analytical purposes. Yet science must employ inductive reasoning in order to generalize and design must use productive inference so as to particularize. These two modes of reasoning can be distinguished by the role the hypothesis plays. The outcome of deductive reasoning is a decomposition which comprises the characteristics of the design that emerge from analysis of the whole composition; and the outcome of inductive reasoning is a supposition, a working rule of some generality – a model.

Such a speculative design cannot be determined logically, because the mode of reasoning involved is essentially abductive. It can only be inferred conditionally upon our state of knowledge and available evidence. Deductive methods can then be used to predict measures of expected performance applicable to the particular design proposal. Concerning the question of value, a design in itself has no value. It assumes relative value through comparison with other designs. As such, evaluation assumes that suppositions about worth, preference, desirability or utility can be inferred. These suppositions form the substance of the productive phase of designing. Thus, the models required to produce design alternatives are value-laden. Therefore, ‘value theory is the essential foundation of any rational theory of design’ (March 1976). As Peirce writes, abduction, or production, is the only logical operation that introduces any new ideas. Induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis. Figure 2-1 shows March’s adaptation of Peirce’s ideas (March 1976).

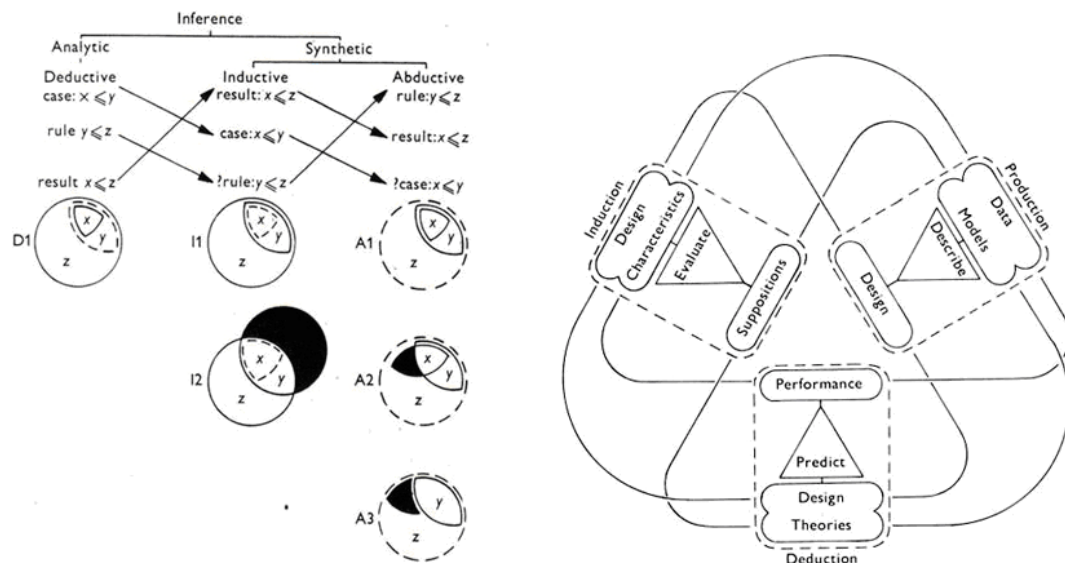


Figure 2-1: Peirce’s modes of inference and March’s PDI-Model

For the team of architect-scientists in Cambridge, the bridges between architecture and other disciplines were firmly established. Upon the concept of modeling designed and described by the Cambridge philosopher Hesse (1966) in her book ‘Models and Analogues in Science’, models, quantitative techniques, structuralism were all up driving the discourse. By 1968, the research group thrived on producing the one model after the other, the one list of equations after the other, and on firmly establishing the usage of mathematics in architecture and planning. In that sense,

mathematics itself defined in the most general sense as a science dealing with definition and manipulation of symbolic models became an indispensable ally to architecture, a science and art dealing equally with definition and manipulation of pictorial and symbolic models.

Still, the focus here is not to give a definite account of this history of formal systems in architecture. Instead one system will be used extensively to provide the scaffolding for the presentation of various general characteristics pertaining to their structure and usage in architecture. This system, Stiny and Gips' 'algorithmic aesthetics', is used here because of its generous structure that a) deals equally with various art forms; b) addresses both analysis and design; c) is built upon both the constructive and evocative modes of understanding; and d) makes extensive usage of the idea of algorithm and computation and therefore formalizes all above issues. All definitions here follow closely Stiny and Gips' conception of aesthetics as the philosophy of both criticism and design in the arts, a definition in itself extending Beardsley's conception of aesthetics as metacriticism, otherwise known as philosophy of criticism (Stiny and Gips 1978). The work here reviews the original model by Stiny and Gips (1978), and two more models associated with it, the 'design machine', a model for design by Stiny and March (1981) and the 'Vitruvian machine' (Economou and Riether 2008), a recent adaptation of the model based on a mapping of the design machine upon the Vitruvian triad (Morgan 1914) in the earliest surviving account of architecture discourse. The review here concludes with several precedents and applications from architecture discourse cast within these systems and especially case studies drawn from applications of group theory in formal analysis and composition in architecture design.

2.2. Formal systems

'Representation that is verbal is classical. By contrast, visual representation is non-classical because of its lack of primitives...Both kinds of representations are interrelated and blurred at the boundaries... The non-classical ones made with things like lines and planes and solids in shapes, and the classical ones made with things like numbers, words, and symbols.' Knight and Stiny (2001)

Formal systems in architecture are concerned with questions about how existing works of architecture can be described, interpreted, and evaluated and with questions about how new works of architecture can be created. A formal system dealing with description, interpretation and

evaluation of an existing work of architecture is called an ‘analysis system’. A formal system dealing with the creation of a new work of architecture is called a ‘synthesis system’ or design system. In this sense formal systems are dealing with questions of criticism and design in architecture, with the foundations of criticism and design in architecture and in essence with the philosophy of criticism and design in architecture.

The medium of all such constructs can be computation. A nice account of computation has been given recently by Knight and Stiny (2001) wherein two aspects of computation, ‘representation’ and ‘process’ are considered as generators of and as species of computation at large; representation has to do with the way objects in a computation are described and process has to do with the rules that are used to carry it out. A basic division of representation in verbal and visual kinds (‘classical’ and ‘non-classical’ vocabularies) and a corresponding division of process in terms of explanation and results (classical processes if the results are understandable in terms of the rules and non-classical if the opposite) produces basically four categories of computation. These categories are then combined under a basic schema of representation/process to produce the following four categories of computation: a) classical/classical computation; b) classical/non-classical computation; c) non-classical/classical computation; and d) non-classical/non-classical computation. The four categories are shown pictorially in Figure 2-2.

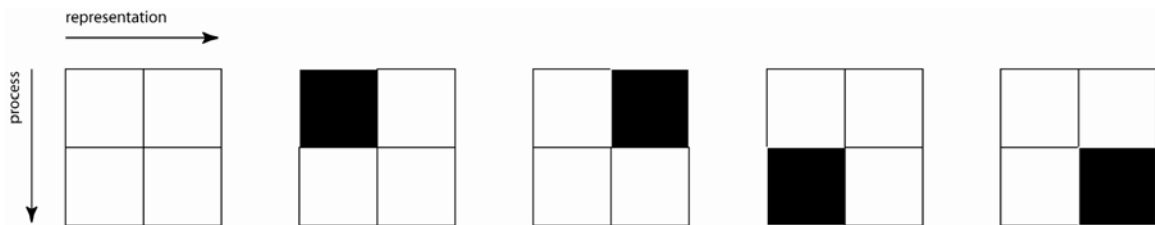


Figure 2-2: Four categories of computation with respect to representation and process

2.3. The structure of formal systems

‘Aesthetics is concerned with questions about how existing works of art can be described, interpreted, and evaluated and with questions about how new works of art can be created. The description, interpretation, and evaluation of an existing work of art is called criticism. The creation of a new work of art is called design.’ Stiny and Gips (1978)

The original schema for algorithmic aesthetics proposed by Stiny (1978) postulates a structure for criticism and design of works of art based on informational process models of thought. The basis of the structure postulated for Stiny’s criticism algorithms and design algorithms is modeled after Kenneth Craig’s model of thought (1943). Craig model consists of three essential properties: a) translation of external processes into symbols (receptor); b) arrival at other symbols by processes of reasoning (theory); and c) retranslation of these symbols into external processes (effector). Figure 2-3 shows the basic structure of Craig’s model of thought.

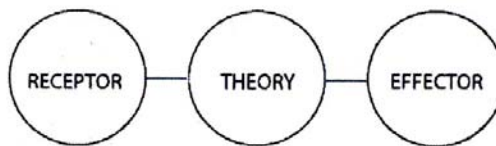


Figure 2-3: A diagrammatic representation of Craig’s model of thought

2.3.1. Aesthetics machine

‘An algorithm is an explicit statement of the sequence of operations needed to perform some task.’ Stiny and Gips (1978)

The basic novelty of Stiny’s model with respect to Craig’s model – essentially an input-output construct – is that it requires a third component to augment the basic structure of computation. This new component is called an ‘aesthetic system’ and is situated in-between the typical input-output schema of Craig’s model of thought (Stiny and Gips 1978). A second powerful novelty of the model is the postulation of an identical structure for criticism and design in arts. In both cases the proposed formal system consists of four components, a receptor (input), an effector (output),

an aesthetic system distinct from both input and output, and an analysis or synthesis algorithm (theory) that uses in different ways the three other parts. Analysis systems and design systems share the same structure. The task of an analysis system is to produce a response to an architecture object as a work of art; the task of a design system is to produce an architecture object as a work of art with respect to some initial conditions. The core of both is the design of the aesthetic system. A diagrammatic representation of the Stiny's model for criticism and design in the arts is shown in Figure 2-4.

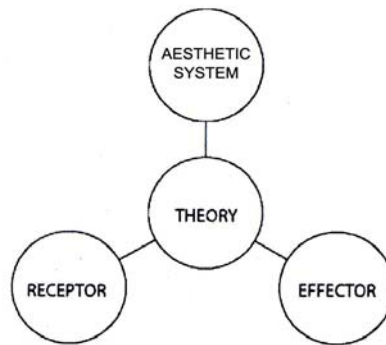


Figure 2-4: A diagrammatic representation of Stiny and Gips' formal model for criticism and design in the arts

The receptor contains a list of descriptions of events, objects or processes of the outside world. Objects and events have an infinity of properties that may be of interest but the ones that are encoded in the receptor are only those that are matching the given requirements and bias of the machine. This list may contain a finite sequence of symbols encoding texts, drawings, images, sounds, numbers, and so forth. The receptor consists of two parts, a transducer and a linked algorithm to encode the output of the transducer into a description consisting of symbols. The transducer can be a television or infrared camera, a microphone, textual survey responses, two-dimensional or three-dimensional scanner, a satellite recorder, and so forth. Less fancy but infinitely more complicated receptors are own personal sensory machinery – eyes, ears, hands, and so forth. The complexity of the structure of the receptor depends on the complexity of the design of the transducer and the linked algorithm. The output of the receptor can be very straightforward as in a bitmap array of color values of a scene or very complex as in a textual description of a scene. Furthermore, the relationship of the external event or process and the description of the receptor cannot be fixed; different receptors may describe the same process in different ways and different processes may be described in a similar way by one receptor.

The effector contains a list of instructions to produce a response to the receptor. The effector consists of two parts, an algorithm to convert a description of design – set of drawings, datasets, texts and so forth– to instructions to produce the result of the computation and a transducer to instantiate the design. The transducer can be a two-dimensional or three-dimensional printer, a two-dimensional or three-dimensional numerically controlled milling machine, a robot to assemble parts, a speaker, and so forth. Less fancy but infinitely more complicated effectors are our own personal motor machinery – hands, legs, muscles, voice and so forth. The complexity of the structure of the effector depends on the complexity of the design of the transducer and the linked algorithm. The output of the effector can be very straightforward as in a printed bitmap of a scene or very complex as in a painterly description of a scene.

The aesthetic module of the formal system is the heart of the whole construct. This system contains a finite sequence of symbols encoding texts, drawings, images, sounds, numbers, and so on, and more specifically it includes descriptions of all possible designs of a certain kind. Each language (set of designs) may be defined in terms of some fixed point of interest, say the Palladio designs, and each may contain diverse descriptions such as three-dimensional models, drawings, or diagrams. Languages may be ordered in any desired degree of complexity defining elaborate structures cutting across spatial and temporal boundaries. Each language may be defined strictly by enumerating the designs in the set or by identifying rules for their generation. The key idea is that aesthetic systems exist independently of other considerations and that their use and value in a computation depends upon the fitness between them and a criticism or design inquiry.

Finally the theory component of the design machine is the link connecting the other three components of the machine; it determines the fit between a design and a design context defined by a receptor and effector. Essentially the theory supplies the principles that enable a design machine to choose the most suitable design for a design context.

2.3.2. Design machine

‘The attempt to formalize things as algorithms leads to a much deeper understanding than if we simply try to understand things in the traditional way.’

Knuth (1973)

The design machine is an adaptation of the aesthetic machine and its goal is the specification of an algorithmic structure for design (Stiny and March 1981). The rules in this system depend on three things: a) the rules given to compose designs, i.e., to construct their descriptions; b) the rules given to describe designs in other terms pertaining to their intended meaning and purpose, or the way they are connected to a complex of associations and ideas; and c) the rules given to assess the quality of designs in terms of the way they are composed or the way their meaning and purpose are described. The original aesthetic module is substituted here from the language of designs and the analysis or design algorithm used respectively for criticism or design is specifically here substituted by the theory module. The diagram of the design machine is shown in two different versions in Figure 2-5; the first emphasizes the main logical connections among components. The second stresses the relationship among components giving an emphasis on the receptor and the effector as the design context of a given design inquiry.

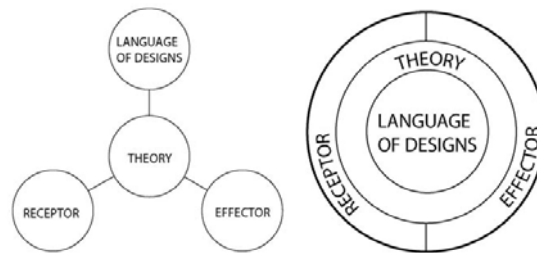


Figure 2-5: Two diagrammatic representations of the Stiny and March's design machine

2.3.3. Vitruvian machine

The Vitruvian machine (Economou and Riether 2008) is a formal model of architecture composition and analysis that partially extends the existing model for criticism and design mentioned so far in two significant ways: a) the model correlates both the aesthetics machine and the design machine with architecture discourse and particularly -and polemically too- with the

earliest surviving treatise on architecture and the three Vitruvian categories of architectural form; and b) the model exemplifies its partition to map with existing architectural discourses and to suggest a generous theoretical framework for analysis and design in architecture discourse.

More specifically, this formal system is mapped upon the earliest model of architecture discourse surviving in the writings of Vitruvius (Morgan 1914) and his account of the three principles of architecture, the categories of ‘venustas’, ‘firmitas’ and ‘utilitas’ - typically translated as beauty, firmness and commodity. These categories of description, interpretation and evaluation of form directly allude to the Aristotelian foundations of this work and the corresponding interpretative framework of architecture in terms of geometric, material and functional characteristics respectively. The mapping between the two models is isomorphic. The receptor is mapped to utilitas (commodity) (U) and to function broadly conceived to include technical specifications, performance specifications, and engineering specifications and so on. The effector is mapped to firmitas (firmness) (F) and to materiality broadly conceived to include all technology specifications and production specifications. The aesthetics module of the aesthetics machine or the language module of the design machine is mapped to venustas (beauty) (V) and to geometry broadly conceived to include all pictorial and spatial descriptions of form. Figure 2-6 shows the diagrammatic representation of the isomorphism between the design machine and the Vitruvian categories and the resultant diagram for architecture design termed here the Vitruvian machine.

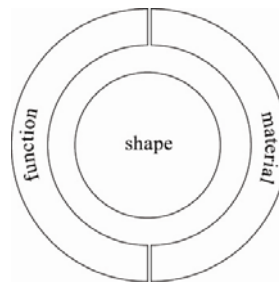


Figure 2-6: A diagrammatic representation of the Vitruvian machine.

This diagram for design suggests a complete structure for alternative definitions of design processes. The possible combinatorial subsets of the Vitruvian triad are $2^3 = 8$ including the empty set that suggests a null input and response, and the possible theoretical constructs for criticism and design are therefore eight. These constructs are nicely mapped to existing discourses of architecture that inform one another. Furthermore, these eight modules can be structured in three sets that correspond to the three subsets of distinct ordinal numbers for the Vitruvian set to

suggest a rising complexity in the discourse of architecture. Excluding the subset of the null input-output, three modules isolate one element of architecture discourse $\{V\}$, $\{U\}$, $\{F\}$, three modules are comprised by two elements $\{V,U\}$, $\{V,F\}$, $\{U,F\}$, and one last module consists of the complete triplet $\{V,U,F\}$. A brief presentation of all eight modules in formal criticism and composition in architectural design follows below.

The null set does not specify any action for criticism or design. This null set with its rather Zen qualities can be mapped to discourses for criticism or design where there are no instructions nonesoever, no deliverables and where everything goes. Figure 2-7 illustrates the null Vitruvian machine.

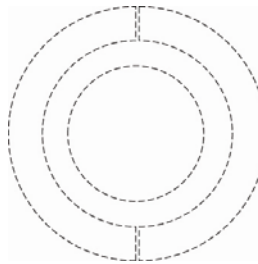


Figure 2-7: A partial Vitruvian machine with no input or output

The second module foregrounds the domain of a geometric vocabulary as the context for formal criticism and design studies. The module foregrounds the history and logic of geometry in the description and construction of space. The formal vocabularies in this module are abstract geometrical terms including points, lines, planes, triangles, squares, circles, conic curves, Bezier curves, NURBS, and so forth, as well as evocative spatial descriptions such as porosity, permeability, balance, symmetry, proportion, order, disorder and so on. Figure 2-8 provides a diagrammatic representation of the Vitruvian machine foregrounding geometry or shape.

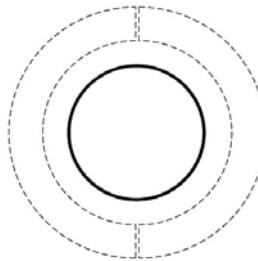


Figure 2-8: A partial Vitruvian machine foregrounding geometry.

The third module foregrounds the domain of a functional vocabulary as the context for formal criticism and design studies. The module foregrounds the history and role of function in the description and construction of space. The formal vocabularies are functional definitions, constraints, conditions, and relations. Figure 2-9 provides a diagrammatic representation of the Vitruvian machine foregrounding function.

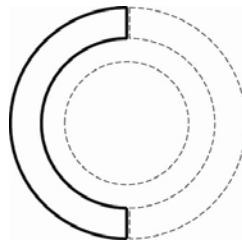


Figure 2-9: A partial Vitruvian machine foregrounding function.

The fourth module foregrounds the domain of material vocabulary as the context for formal criticism and design studies. The materiality of form - hard, soft, elastic, rough, smooth, opaque, transparent, translucent, and so forth, supports, enables foregrounds or even contradicts the criticism and the design description. This module explores the affordability of a variety of different materials to render diverse possibilities of criticism and design and suggests a systematic exploration of materiality and fabrication methods. Figure 2-10 provides a diagrammatic representation of the Vitruvian machine foregrounding materiality.

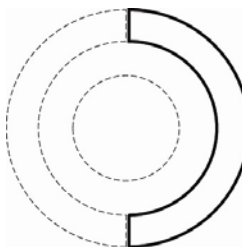


Figure 2-10: A partial Vitruvian machine foregrounding materiality

The fifth module pairs function and materiality and proposes a loop of formal criticisms and design explorations informed by functional and material considerations. Some functional

arrangements are specifically enabled by specific materials and some materials afford different functional organizations to emerge. The loop between the two domains suggests two different trajectories of reasoning, one starting from function and testing against materiality (UF) and the other way around (FU). The partition is designed to explore the interrelation of programmatic organizations and material properties without taking into account geometrical or shape considerations. Figure 2-11 provides a diagrammatic representation of the Vitruvian machine foregrounding function and materiality.

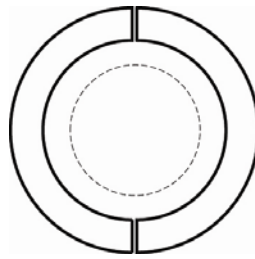


Figure 2-11: A partial Vitruvian machine foregrounding function and materiality.

The sixth module pairs function and geometry and proposes a loop of formal criticism and design explorations informed by functional and geometrical considerations. Some spatial arrangements are apt to allow specific functions and some functions often emerge in specific spatial organizations. The loop between the two domains suggests two different trajectories of reasoning, one starting from function and testing against geometry (UV) and the other way around (VU). The partition is designed to explore the interrelation of programmatic organizations and formal languages without taking into account material or construction considerations. Figure 2-12 provides a diagrammatic representation of the Vitruvian machine foregrounding function and geometry.

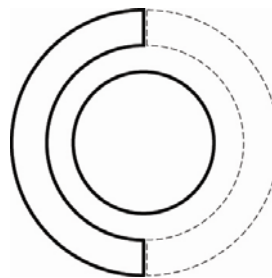


Figure 2-12: A partial Vitruvian machine foregrounding function and geometry.

The seventh module pairs geometry and materiality and proposes a loop of formal criticism and design explorations informed by geometrical and material considerations. Some systems of geometry are informed by specific models of construction and some construction techniques are developed to meet geometric demands. The loop between the two domains here suggests two different trajectories of reasoning, one starting from geometry and testing against materiality (VF) and the other way around (FV). The partition is designed to explore the interrelation of design specifications and fabrication methods, including aspects of prototype structures, form and formwork, scalability and so forth. Figure 2-13 provides a diagrammatic representation of the Vitruvian machine foregrounding geometry and materiality.

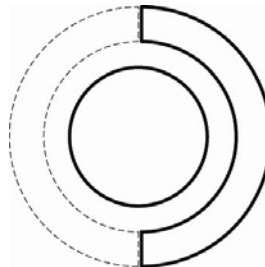


Figure 2-13: A partial Vitruvian machine foregrounding geometry and materiality.

The eighth exercise closes the design inquiry suggested by the Vitruvian machine and fully engages all three aspects of the model. The loop between the three domains here suggests $3! = 6$ different trajectories of reasoning; VUF, VFU, FVU, FUV, UVF, and UFV. Every theoretical trajectory selected has to be understood, reflected and critiqued upon the ways it informs and it is informed by the other theoretical trajectories of the design process. And still, all trajectories should be present in the end suggesting a totality and complexity that resent unpacking and command alternative interpretative discourses. The partition is designed to allow for a full immersion in architecture discourse with complete sets of programmatic requirements, performance specifications, technical specifications, engineering specifications, production specifications and so forth. Figure 2-14 shows the complete Vitruvian machine.

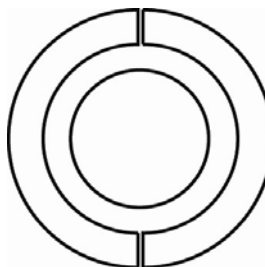


Figure 2-14: A complete Vitruvian machine.

2.4. Analysis and design systems

One of the central claims of the original aesthetics machine is its applicability in both criticism and design. The formal systems that can be constructed upon this function are a) the analysis systems; and b) the design systems. A formal system dealing with analysis requires as an input component a description of an architecture object and, as output component, a statement about some formal properties such as type, arrangement, symmetry, rhythm, proportion as so on. Analogously, a formal system dealing with design requires, as input component, some rules or data with schemata that particularize these data and as output component a description of an architecture object. In both systems, interpretations are defined independently of actual architecture objects; descriptions of architecture objects are manipulated and co-related to give an interpretation of these objects in terms of associations or constructive rules.

2.4.1. Analysis systems

An analysis system has, as an input component, a description λ of the object and, as an output component, a statement about some formal properties β such as symmetry, proportion, balance, or rhythm. This formal system is generally described by an algorithm of the form $\langle \lambda, \beta \rangle$ wherein λ is the description of an object and β is the list of evocations generated by the description λ ; this algorithm specifies how an object with description λ is understood by listing the properties β . In general, in a system of this type the input component might be a list of descriptions of a building, including pictorial data such as plans, sections, elevations, sketches or photos or symbolic data such as texts, tables or any other form of symbolic analyses, and the output component might be a statement about structure, arrangement, rhythm, or symmetry. A series of examples that can be modeled in this fashion follow below.

An example of this system in design is March and Steadman's (1971) approach in the analysis of houses by Frank Lloyd Wright's. Widely diverse designs are topologically equivalent and share the same underlying structure; topological transformations of the geometry of three houses which is generated by repeated applications of different geometric units. Units are composed of an equilateral triangle, a square and a circle that produce three individual designs which share the

same underlying structure. Figure 2-15 shows the plans and the underlying graph envisioning the correspondence of the various subspaces in the houses.

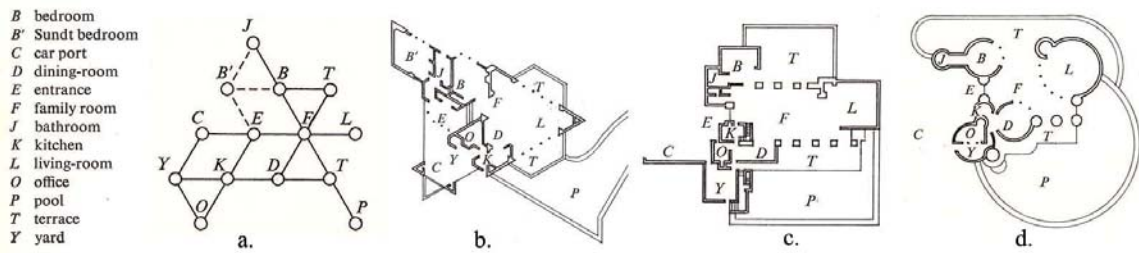


Figure 2-15: March and Steadman's topological equivalencies of F. L. Wright houses.
a) graph; b) Sundt; c) Life; d) Jester

It is quite interesting that this specific formal analysis provided the blueprint for a host of other similar types of formal analyses that all sought to exhibit the common transformational structure that links various types of design. The very same formal analysis as above is repeated by Laseau (1992) in other case studies including the Life house, the Hanna house, and all the variations of the Lewis house and the Jacobs house by of F.L. Wright and is given in Figure 2-16.

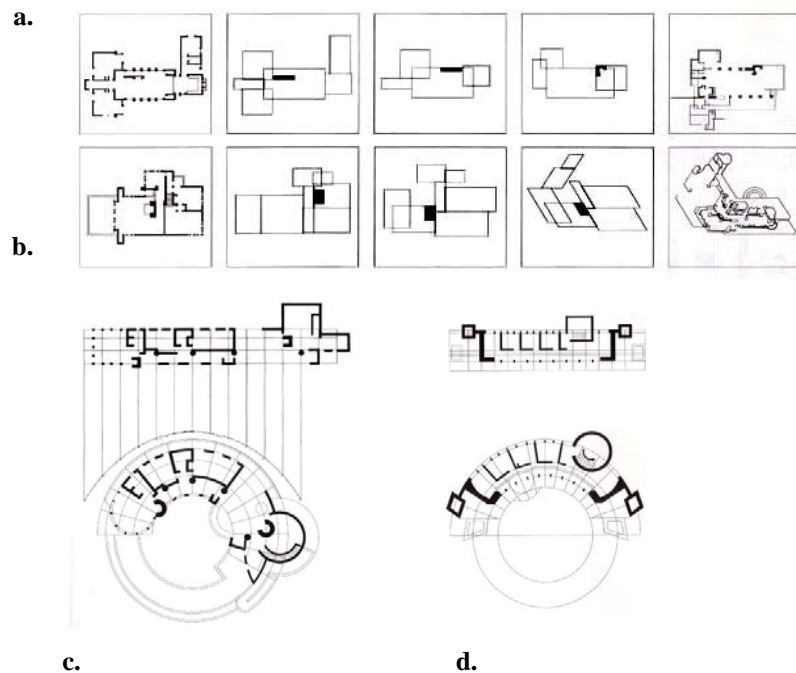


Figure 2-16: Laseau's transformational equivalencies of F.L Wright houses.
a) Life; b) Hanna; c) Lewis; and d) Jacobs

Another example in formal analysis is March's analysis of the ratios in R.M Schindler's How House in Los Angeles, California. The partition of the whole plan or various parts of the house, such as the piano nobile or the maid's wing, can be seen as a straightforward recursive application of a family of rectangle of specific ratios associated with music discourse (March 1993). The main construct used in his analysis is the fact that any rectangle characterized by these musical ratios can be divided into rectangles with corresponding musical ratios. A sequence of partitioning of the maid's wing suggests a musical development of the plan as in Figure 2-17.

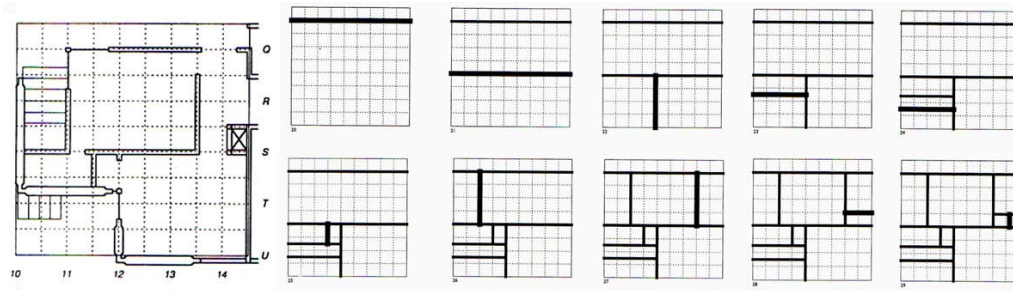


Figure 2-17: March's successive partitions of the maid wing of the How House according to musical ratios

A very different but exciting example is Birkoff's (1933) attempt to commensurate the aesthetic value of form – in fact any form. The ensuing formula (see below), arresting in its simplicity, imposes a basic analogical relationship between the characteristics of order (O) and complexity (C) and introduces various other parameters to address idiosyncrasies of various modalities of form. The general form of the aesthetic measure (M) is given in (1).

$$M = \frac{O}{C} \quad (1)$$

For example, for the case of the rectangle the various parameters that enter the computation are vertical symmetry (V), equilibrium (E), rotational symmetry (R), relation to horizontal and vertical network (HV) and unsatisfactory form (F). The aesthetic measure (M) of the polygons for specific relation among these parameters is then given in (2).

$$M = \frac{O}{C} = \frac{V + E + R + HV - F}{C} \quad (2)$$

The actual computation of these parameters provides an ordering scheme for the arithmetical ratios of similarly positioned rectangles and provides a specific framework of interpretation of rectangular forms suitable for specific types of composition. It is worth noting that Birkhoff (1933) drew upon the experiments of the psychologist Fechner (1860), (1876) who ascertained that the most satisfactory series of rectangular shapes, including the square, is the sequence within the range of one-to-one and one-to-two presented in Figure 2-18.

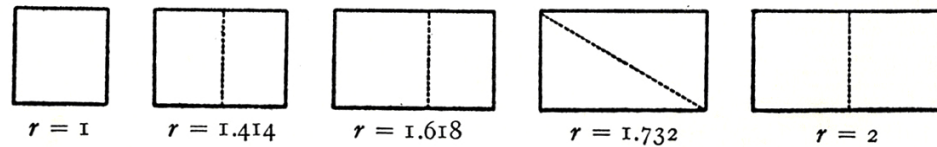


Figure 2-18: Five rectangles with ordered ratios r by Birkhoff's aesthetic measure.

2.4.2. *Design systems*

A design system has, as an input component information α needed to construct an object and as an output component a description λ of an object. This spatial system is generally described by an algorithm of the form $\langle \alpha, \lambda \rangle$ whereas α is the information needed to construct the object and λ is the description of an object; the algorithm specifies how an object is understood by listing the information α that generate the description λ of an object. The information given by the input may be considered as a list of instructions to be followed or as a list of data to be acted upon. In the first case the rules provide all the necessary information to construct the object; and in the second case the data have to be acted upon by a schema encoded within the system.

The input component is considered as a list of instructions when it entails an explicit provision of primitive elements and rules for the combination or organization of the elements; in this case the rules are applied to the elements and result in the description of the object. An example of this formal system in spatial design is Froebel's 'kindergarten method' for the construction of simple designs using a series of geometrical 'gifts' and a system of 'categories' of geometrical forms (Stiny 1980). In this pedagogical system, a series of simple geometrical shapes are given to the children along with some rules of combinations to create designs defined in a system of categories.

In general, in a formal system of this type the input component might be any list of primitive spatial elements and a list of rules that specifies how the parts are combined; the output

component would be the description of a building in plan, elevation, section or any three-dimensional perspective view that conforms to the imposed rules. In a formal system of this type dealing with the form of music, the input component might be a simple motive and the rules for generating the piece from that motive, and the output component would be the score of a piece or any other description of a piece.

The input component is considered as data when it gives explicitly the primitive elements that are developed or arranged according to a schema for a large class of descriptions of a certain type. An example of a system of this type in spatial design is Durer's schema for the description of the human face. In this system each individual face is a 'parametric transformation' of a standard schema; the data particularize the proportions of a dimensionless grid and produce descriptions of different faces which all fit the schema.

Examples of design systems for which the input component is a list of primitive spatial elements and a list of rules that specifies how the parts are combined have been nicely captured by the shape grammar formalism (Stiny and Gips 1978); Stiny (1976); (1985); (1990); (1991); (1992); (Knight 1994) and especially by the 'kindergarten grammars' (Stiny 1980). The latter is a type of spatial algorithms that formalizes the pedagogical character of Froebel's kindergarten method and extends the notion of construction of languages of designs (sets of shapes) from scratch (Stiny 1981); Knight (1992) (1994). As already stated earlier, this formal system uses a series of geometrical 'gifts' and a system of 'categories' of geometrical forms and it is a formalization of a pedagogical system for the training of the children invented by Frederick Froebel, wherein a series of simple geometrical shapes are given to the children along with some rules of combinations to create designs defined in a system of categories. Figure 2-19 shows a kindergarten grammar consisting of simple shape, one labeled rule and a design in the pre-specified language.

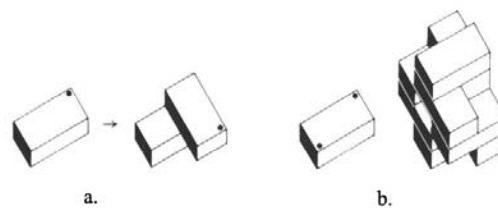


Figure 2-19: Stiny's simple shape grammar consisting of one initial shape and one shape rule.

An example of data that parameterize a specific schema is Sullivan's approach to design. This approach can be embedded in the web of his ideas on functionalism. For him, design includes

functions that satisfy cultural and higher spiritual necessities of humankind, and not just the utilitarian needs for which ‘form follows function’ has mistakenly been cited (Sullivan 1922). Figure 2-20 shows Sullivan’s generation of a series of motifs based on the structure of the square.

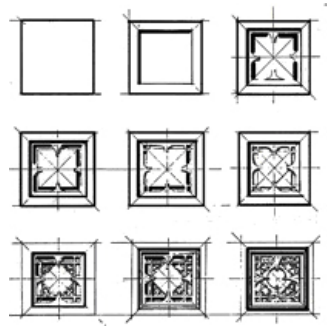


Figure 2-20: Sullivan’s square motifs

2.5. Constructive and evocative systems

‘A basic underlying assumption of this study is that there is no single, correct way to describe, interpret, and evaluate any given object as a work of art. Which objects are considered works of art and how these objects are understood and evaluated as works of art is purely a matter of convention.’ Stiny and Gips (1978)

A second central claim of the original aesthetics machine is that analysis and design in aesthetics use two different models of understanding that each suggests a profoundly different world-making. More specifically, it is suggested that aesthetic systems can be characterized and computed in terms of their interpretations. Typically objects can be understood or interpreted in terms of a) how they can be constructed, and b) what associations, ideas and emotions they evoke. The former systems are referred to as ‘constructive systems’ and the latter as ‘evocative systems’. Any other system of interpretation can be based on any combination of these two basic types.

The fundamental formal distinction between the two systems is that the description of the object λ is the output component of the computation in the constructive systems, whereas in the evocative systems, it is the input component. A typical example of a constructive system is the understanding of a number sequence in terms of the rules used to generate it. An interpretation of an object has the form $\langle \alpha, \lambda \rangle$, where α is the list of rules to produce the description λ or a schema for a large class of descriptions of a certain type. A typical example of an evocative

system is the understanding of a number sequence. It is like a telephone number and a corresponding list of associations involving the person with this telephone number. An interpretation of an object has the form $\langle \alpha, \lambda \rangle$, where α is the list of rules to produce the description λ or a schema for a large class of descriptions of a certain type.

This distinction between constructive and evocative understanding can be used in both analysis and design systems to produce basically four different structures for criticism and design: a) a constructive – analysis system; b) an evocative – analysis system; c) a constructive – design system; and d) an evocative – design system. A series of applications based on these formal systems are presented below.

2.5.1. *Constructive – Analysis systems*

'Advances in modern science and technology have resulted from the application of less familiar mathematical models: groups, rings, fields, vector spaces, linear and Boolean algebras; topology, graph theory, and variety of algebraic geometry; linear, nonlinear, dynamic, and Boolean programming.' March (1972)

A series of constructive spatial systems is presented here using tools from set theory, group theory, graph theory, Boolean algebra, permutations and shape grammars. All systems entail some mathematical model involving a class of undefined elements and relations between these. All these models reproduce suitable chosen features of the physical situation if it is possible to establish rules of correspondence between specific environmental elements and corresponding mathematical elements and relations in the models, that is, if it is possible to have what is technically known as an isomorphism between the two domains.

A system relating the organization of space with graph theory to produce generic house plans has been given by Steadman (1971). The model suggests that it is possible to determine the most probable distribution of spaces independently of any particular arrangements. Figure 2-21 shows the isomorphism between graph theory and room layout.

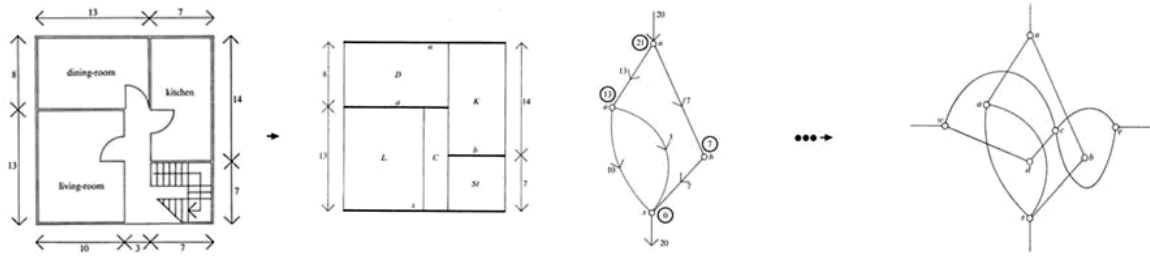


Figure 2-21: Steadman's graph theoretical descriptions of generic housing plans

A different application of graph theory, using models from the theory of electrical networks, produces a graph that represents the adjacencies of the relative positions of the rooms in the plan, and their exact dimensions and shapes (Bullock 1971). Figure 2-22 shows an illustration from the original model: by translating the course options into a switching circuit, a loose fit approach can be tailored to limit the space and time to accommodate the course requirements.

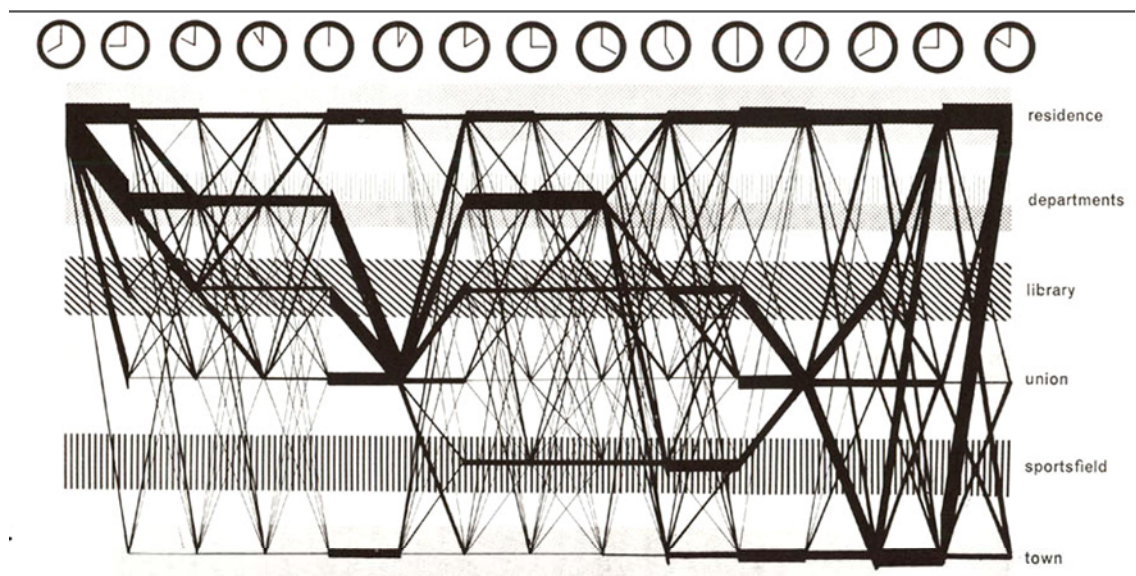


Figure 2-22: Bullock's use of electrical networks to model activities

Applications of probability models are numerous. A probability assignment is a numerical encoding of a state of knowledge. The rationale of this approach is straightforward: a) Write down as much as you know about the system of interest, encoding this knowledge in a set of equations; b) infer, mathematically, the most likely state of the system on the basis of the given information while maximizing your impartiality; c) Compare your inferences with your

observations. Differences between observed and expected represent either what you do not know, or what you know but have failed to make explicit. Generalization is inductive and it consists in perceiving possible general laws in the circumstances of special cases.

A nice spatial system using this construct is a zoological study of cells by Weiss (1955). Figure 2-23 shows a diagram from the application of the model representing a system of order defined by the linear distribution of a three-by-three magic square numbers mapped into a twenty-five-square grid. The cells symbolize the range within which a dot inside is free to roam – a range that has nine degrees of freedom – responsibility to move within a constraint orbit.

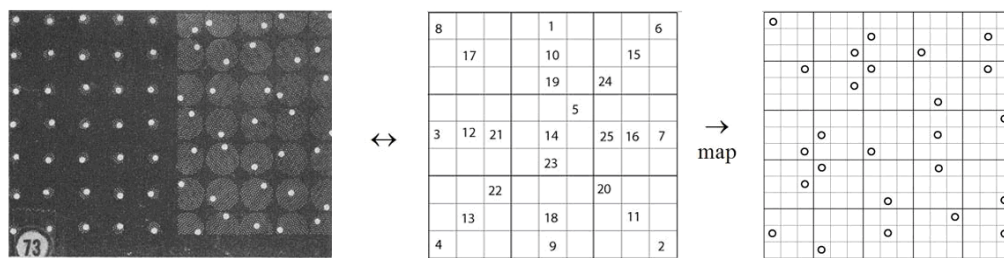


Figure 2-23: Application of the rule of order based on Weiss's probabilistic model

Another example of such models is Conway's game of life where interaction between cells is based on adjacency requirements (Wolfram 1984). An illustration of such interaction typically modeled by the mathematics of cellular automata is given in Figure 2-24. A cellular automaton is an algorithm for generating a set of cells given a prior set of cells (Wolfram 1984). An isomorphism between one-dimensional and two-dimensional cellular automata and settlement patterns in Africa has been recorded by (Eglash 2005).

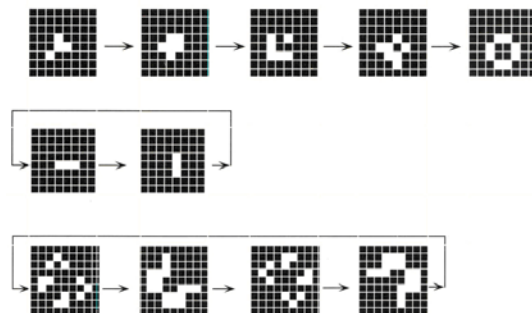


Figure 2-24: Conway's game of life

A different system using tools from Boolean algebra in architectural design is March's (1976) Boolean description of built form encoded into his minimal representation schemes. In this type

of representation a plan is mapped upon a dimensionless grid whose combinations of empty and filled cells signify the arrangement of spaces or the arrangement of walls and so on. For example this representation allowed March (1976) to argue that the Mies' Brick house, the Gropius' Dessau building and the Schindler's King's Road house appeared to share a similar structure of a butterfly motif with extending wings, and more specifically that the Mies' house and the Schindler house shared an identical Boolean representation – March's Lectures Notes (Economou 1992). Figure 2-25 shows the original plans and their minimal representations.

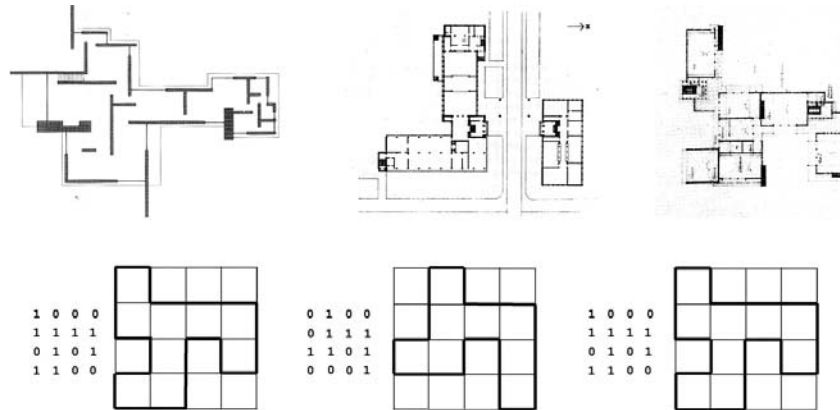


Figure 2-25: March's Boolean descriptions of built form
a) Brick House; b) Dessau building; c) King's Road House

A similar example of a formal description utilizing tools and methods of set theory in architectural design is March's (1971) generation of the ground plan of the 'maison minimum'. Figure 2-26 shows the derivation of the overall plan; walls and doorways are shown in plan and axonometric projection.

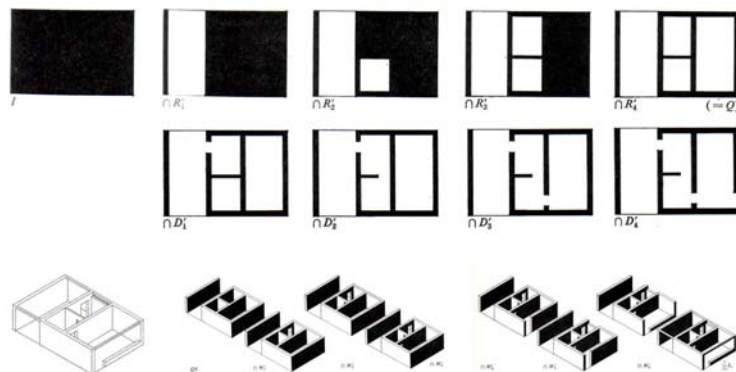


Figure 2-26: Set theoretical generation of the 'maison minimum'

Another formal constructive system is the Galois representation and the Hasse diagrams that both use aspects of formal ordering to classify objects and properties. In this representation design objects are represented as elements of a set and relations and/or operations on these elements and the result is derived from an algebraic model of that design. A rather involved but rich example of usage of partial order lattices and group theory is Park's (2000) representation of the Free Public Library by R. M. Schindler, Jersey City, 1920 with all its sub-shapes of the first-floor plan partially ordered with the full symmetry of a square. Figure 2-27 illustrates how the lattice of sub-symmetries of the square of the Library is constructed.

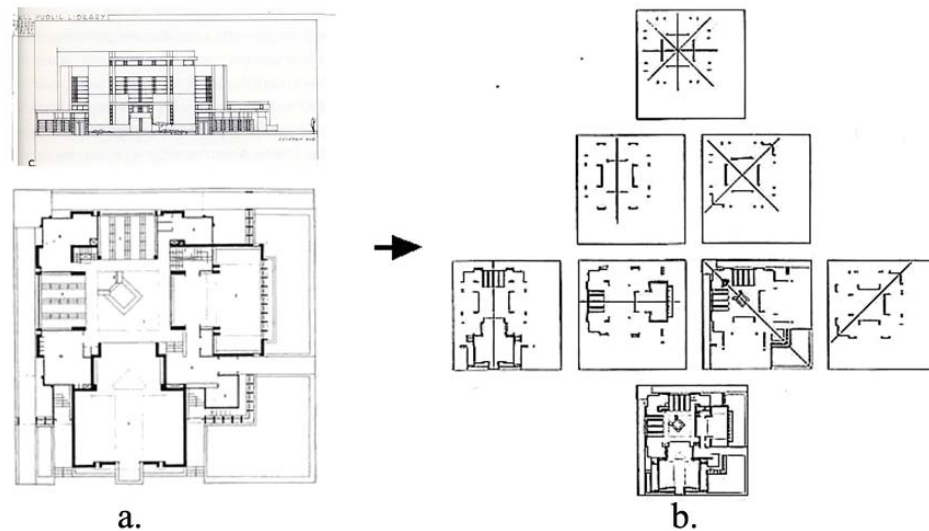


Figure 2-27: Park's lattice representation of the sub-symmetries of Schindler's Free Public library - a) plan and elevation; b) semi-lattice of sub-symmetries

2.5.2. Constructive – Design systems

'Using elementary set theoretical notions, together with simple proportional structures, I found that I could compose in my head quite elaborate works, develop them, and construct related series under various transformations and permutations.'
 March (1972)

An application of set theory in formal composition is March's (1966) serial art where the operations of union and intersection on one basic generator define the whole sequence of spatial motifs. The whole series of arrangement of the structural row and its variations based on a unit and its inverse, their unions and intersection as well as their transformational relations through

reflection and rotation are shown in Figure 2-28. The composition may have been sparked by Albers' square frames.

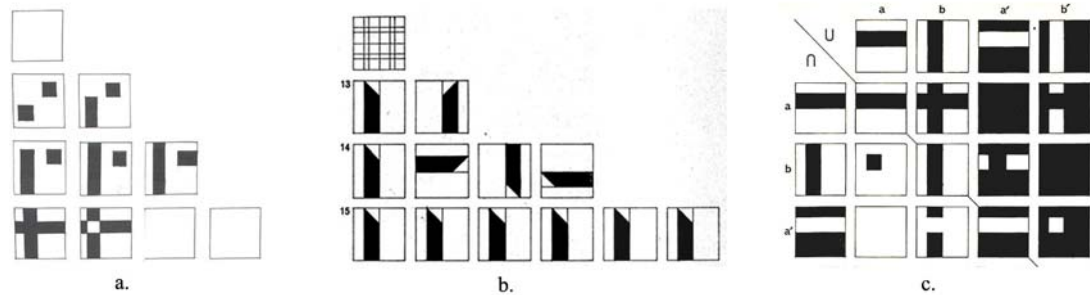


Figure 2-28: Two compositions – a) Albers' square frames; b-c) March's set theoretical composition based on rigid motions and union - intersection operations

Another extension of the graph theoretical representation that captures order in design has been given by Economou for the construction of original classes of design using group theoretical tools and three-dimensional spatial elements – Economou (2001); (2007); (2008). Figure 2-29 shows the generation of a three-dimensional study that consists of the sum of all possible ten symmetry subgroups found within the structure of the dihedral group of order four; details about symmetry groups and their constructs are given later in this chapter as well as in chapter three.

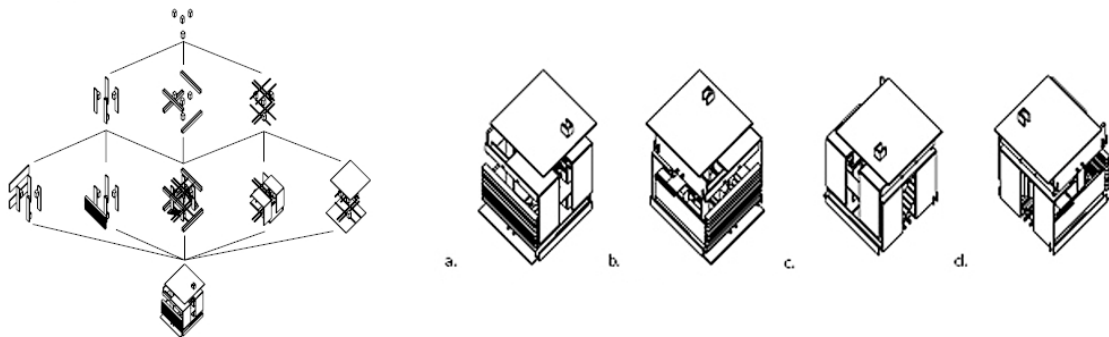


Figure 2-29: A generative description of a three-dimensional D_4 house.

2.5.3. *Evocative – Analysis systems*

'... A mask operated as a palimpsest mysteriously guiding the location of the walls. The space is thus a dislocation induced by the forms of the masks. The effect is to create within the rational space of the grid a violent juxtaposition of perplexing

spaces. The villa Stein at Garches was then considered as the prototype of modern architecture. Rather than a simple fetish, the mask here served as subversion for the order of reason through its spatial implications.’ Tschumi (1976)

Evocative systems are integral components of the aesthetic machine and are used both for analysis and synthesis systems. Various case studies are briefly presented below that belong in both domains and point either to a criticism of existing architecture or to the making of new.

An interesting evocative approach is the one described by Giedion (1954) in his ‘Space, Time and Architecture’ whereas he considers formation of space as one out of two archetypal house-forms: mass and wall. From these two categories all systems receive their basic ordering. Mass systems must be external representations of an internal volumetric order. They convey the conception of a generic solid that has been eroded or cut away as in the Villa Moissi (Flemming 1978). However, the mass can also be thought of as having been juxtaposed with a series of volumetric planes as in Le Corbusier’s Villa Stein (Eisenman 1963). To achieve any systemic organization the vocabulary (volume, mass, surface and movement) must be ordered and therefore clarified by a grammar. Since volume is that property of generic form put forward as being fundamental to any architectural expression, some form of volumetric ordering will occur in any system, without necessarily providing the basis for that system. Moreover, in situations where a volumetric order is dominant, this order can either be continuous or static: both applicable to either a centroidal or a linear situation. There is a third type of volumetric order which is conceived of as a series of volumetric planes.

First, in a continuous system, the movement or circulation is interconnected with the volumetric ordering. The continuous system is associated with early modern avant-garde architecture. Thus we find examples of continuous volumetric ordering in the De Stijl house projects, and the early Cubist and Purist exercises. In De Stijl, the volumetric system is combined with a surface of planar system while in Purist work the volumetric ordering is related to the system movement. Second, in a static system, each volume is expressed or articulated as an individual entity. A sense of total organism is achieved by means of a sequential progression: volumes are linked together as beads on a string. Third, a series of vertical volumetric planes has three recognizable subtypes of volumetric systems. These subtypes depend upon a surface ordering for their definition; the essential characteristic is the volumetric ordering that relates to a surface or juxtaposition of surfaces for its reference. The first subtype is a series of vertical volumetric

planes tensioned from a vertical reference, illustrated by Le Corbusier's Garches. The composition is ordered by a series of volumetric planes defined initially by the front façade. The second subtype, a series of horizontal volumetric planes, is defined by a sequence of horizontal surfaces, tensioned usually from an articulated floor or roof plane: the prototype of this being the Maison Domino. The third subtype of mass-surface system is the volumetric plaid which derives its order from two adjacent surfaces: one horizontal and one vertical plane in juxtaposition. The villa Shodhan by LC combines the principles of the Maison Domino with those of Garches to produce the resulting plaid.

An example that nicely illustrates this spatial distinction is Flemming's (1990) interpretation of Loos' architectural vocabulary as a playful composition of these fundamental properties of volume, mass, and surface. A cubic volume is carved out of its mass by hollowing out interior space. Subtractive operations are used to carve into the surfaces of the remaining walls, creating profiles, and finally windows and doors. Figure 2-30 shows Flemming's interpretation of Loos' Villa Moissi at Lido as a case study of mass and surface architecture.

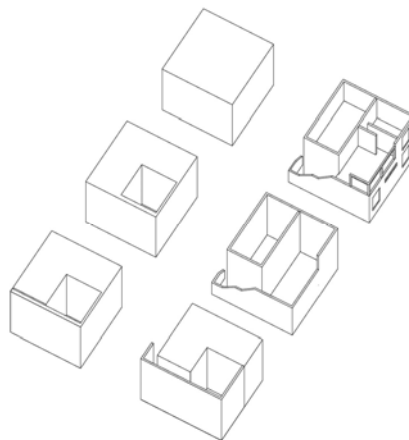


Figure 2-30: Loos' Villa Moissi at Lido: mass and surface architecture

A different form of evocative abstraction occurs when mass and surface are combined to define a plane or layer that provides a datum for organizing a composition. Rowe (1972) asserts that parallel layers organize buildings such as Le Corbusier's Villa Stein at Garches. They establish links with the principles of Purist Aesthetic. Whereas Mies assembles freestanding elements in empty space, Le Corbusier carves its voids out of the solid square frame inwards (Padovan 2002). A variation of the vocabulary of this language contains another basic sub-type: columns and beams within a structural frame and infill elements that generate enclosures and spatial divisions. The frame establishes the whole and divides it into clearly related parts with some logic for

placement of spatial divisions. As it is exposed at the inside, it affirms overall ordering function, and continuity with the outside. Figure 2-31 shows both Eisenman's (1963) and Kulic's (1999) interpretation of Corbusier's Villa Stein at Garches as an example of layered architecture.

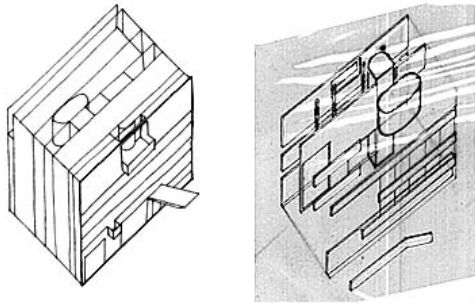


Figure 2-31: Villa Stein at Garches and Layer stratification – a) Eisenman; b) Kulic

This process of formal de-familiarization brings the object into the sphere of new perception. De Stijl too follows this path in the creation of new object world. Objects are described by pure formal relationships derived from formal universals based on geometric abstraction. Figure 2-32 shows two different sets of images, one on abstractions of African heads (Din 2003) and the second on an abstraction of an organic form to geometric form with the case study of the cow by Van Doesburg (Zimmer 2003).

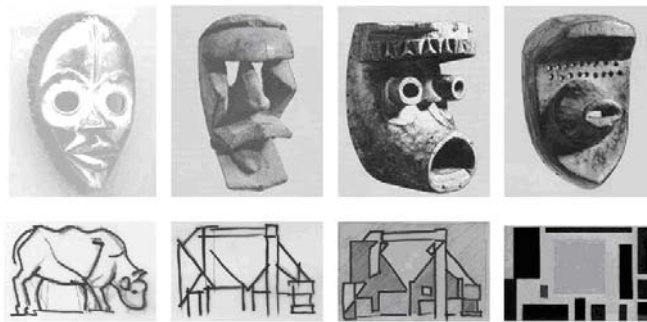


Figure 2-32: Abstraction as an evocative system - African's head and van Doesburg's cow

The evocative distinctions need not be restrained in orthogonal geometrical systems. An architecture of forms that is geometrically non-Euclidean, presents a great opportunity of sculptures that become buildings despite the complex negotiations among figure, structure, and program. The introduction of such amorphous figures into architecture provokes new connections with conventional representation, technique, and context. There has been an extra-Euclidean counter-tradition that has managed to survive through history. It shows up in ancient knowledge

at Sheba-Sirwah (temple of the moon, 1000BC) (Clapp 2001), and Zimbabwe (Imba Huru temple, 1400AD) (Walton 1953) and Yoruba Compound (Fassassi 1978), as well as in modern twentieth century architecture such as in Philharmonie (Sharoun 1995), the German Embassy in Brazilia (Syring 2004), the Ronchamp Chapel (Clark 2005) and the Ulm Building (Meier 1991) and so forth. A series of buildings that foreground and at the same time resist their interpretation in terms of spatial and planar layering is shown in Figure 2-33.

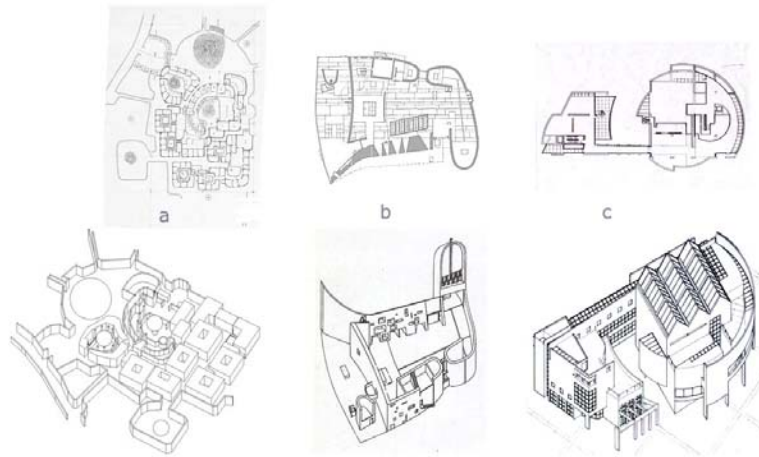


Figure 2-33: Three sculpture-buildings: a) Yoruba; b) Le Corbusier; c) Meier

Collage is either a device for assembling a picture from diverse fragments of discarded materials, or the result of a process of experimentation, notation, and operations. Any collage emphasizes the unity between formal logic and pictorial composition. As a device, literal collage involves the juxtaposition of physical material, whereas phenomenal collage involves the ambiguity and reciprocity of figure/field (Hildner 1997). Figure 2-34 shows a) a free-hand pencil drawing that deconstructs the Gris' painted collage maps the white space-defining fragments; b) a cut-figure shadow image of the 'Guitar Player in Profile' that exemplifies the Cezannesque organizing principle of interlocking cut figures; c) the form of the interlocking with a Woman Listener locked to the Guitar Man along the fault-line for shallow-space | deep-space.



Figure 2-34: Collage as an evocative system. Still Life with Guitar, Juan Gris 1917

Aalto (1940) calls this mode of operation: ‘the Purist overemphatic rhythm in which all impulses run parallel’. This quality of imposed order endows a transparent organization of the capacity of multiple readings of the interconnections between the parts of a whole system. When the figures are endowed with transparency, they interpenetrate without an optical destruction of each other. Transparency implies more than an optical characteristics, it implies a broader spatial order. Transparency means a simultaneous perception of different spatial locations and it may be an inherent quality of substance or it may be an inherent quality of organization. Figure 2-35 shows two examples of layering process.

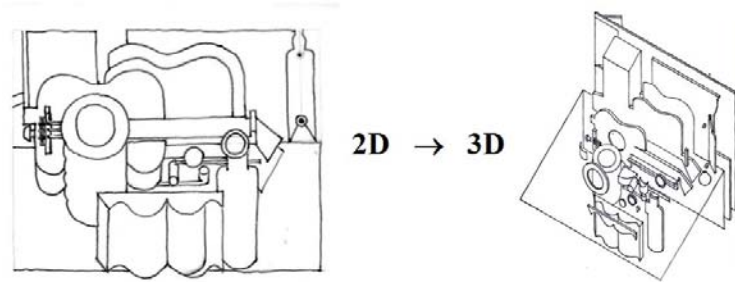


Figure 2-35: Layering as an evocative system. a) Jeanneret's Still-life; b) Hoesli's decomposition

2.5.4. *Evocative – Design systems*

Russian constructivism provides an incredible array of case studies that all foreground a host of diverse aspects of formal characteristics in their construction or associations (Khan-Magomedov 1987). Malevich's ‘suprematism’ and El Lissitzky's ‘prouns’ amplify the cubist measures of portraying space, and suggest separate avenues for manipulating and heightening the spatial effects of abstract form, as does Rodchenko's spatial compositions with line and plane (Senkevitch 1983). Like the Cubists, Malevich (1917) seeks to return to the pristine elements of form, shapes such as squares, rectangles, circles, and triangles. Unlike the Cubists, he reduces the number of elements, increases their size, and eliminates all traces of representation, creating nonobjective compositions as shown in Figure 2-36a.

Emerging as a logical consequence of Malevich's work, the Prouns of El Lissitzky's bridge painting to architecture by conveying explicit spatial depth through three-dimensional renderings of entities imbued with architectonic clarity. Not only do elements of Lissitzky's composition move back and forth, as do those of Malevich's paintings, but they convey a dynamic concept of

kinetic form and space as shown in Figure 2-36b. Thus, the notion of simultaneous projection and penetration in irrational space and the concept of expressing form and space through the rhythmic grouping of elements in imaginary space is analogous to the three-dimensional space to be formulated as a conceptual matrix for their spatial form (Lissitzky 1968).

Drawing upon predecessors, Rodchenko creates a system of pictorial construction in which the process of manipulating elements, rather than the shapes themselves, becomes the focus of perceptual activity (Senkevitch 1983). Rodchenko's compositions for the first time create the definite impression of being constructed rather than composed and so, give birth to Constructivist aesthetic as shown in Figure 2-36c. 'Each line, in itself, neither carries any particular esthetic impact nor implies any pictorial space. And yet, spatial effects of considerable power and elaboration are achieved through the interaction of lines to create transparent planes possessing visual density, scale, and tangible spatial definition' (Senkevitch 1983). A series of suprematist constructions are shown below.

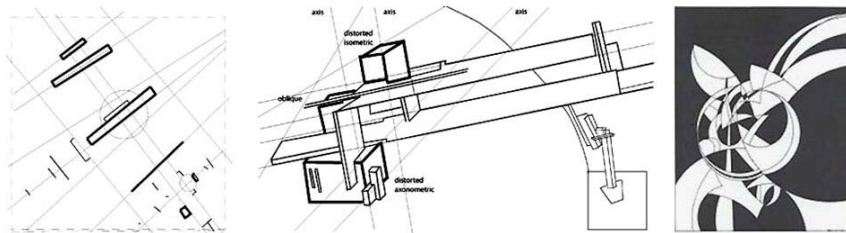


Figure 2-36: Spatial form in suprematism. a) Suprematist drawing (1917); b) Proun 1A (1919); c) Artist's compass (1915)

Typical cases of evocative systems include all the systems that associate forms between different symbolic systems and especially those between architecture and painting. In modern painting, there are two methods to work out physical form: abstraction and collage. Leger's (1972) and Kandinsky's (1979) work shows how forms of abstraction de-familiarize everyday objects through the manipulation of geometry. Figure 2-37 shows how the human form used as a plastic element helps to simplify the geometric order and how the inner relation between a complex of straight lines and a curve is achieved in the work of the 'Mechanical element' by Leger (1972) and the 'Black triangle' by Kandinsky (1979).

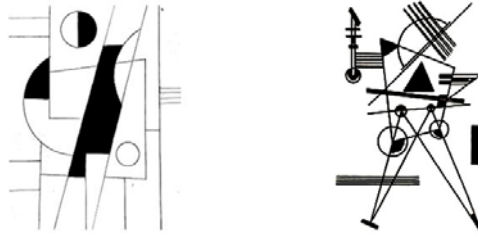


Figure 2-37: Mechanical element and Black triangle

Another archetypal house-form is the study of walls as independent elements considered as pure form and analyzed by Padovan (2002). This leads to neo-plasticism in which rectangular panels are placed in the three major directions to form an abstract composition that follows some ordering principles. The basic element manipulated is a rectangular panel placed vertically or horizontally, frontally or across. Windows and other openings are given by the gaps between panels. To close the gaps, a second, transparent panel is needed. Van Doesburg's Counter-Construction of Eesteren's Private House (1923) is a case study that foregrounds planarity of spatial elements as the principal organization of form. Van Doesburg explores the implications of the Eesteren house design in a series of analytical counter-constructions in which the solid volumes are isolated in disjoint classes: vertical planes – panels, horizontal planes – slabs, and cubic volumes – collage elements. This technique to isolate the elements of composition is an unequivocal example of syntactic layering and an example of surface ordering. Rules must take into account both ways of operating and show first how new panels are added, then the insertion of transparent panels once solid panels have been placed and finally that of linear and volumetric elements. An extension of these rules can set the generation of truly three-dimensional compositions with the use of functionally equivalent interchangeable parts. Figure 2-38 shows Padovan's reproduction of Doesburg's counter-construction from Eesteren project as an example of surface architecture.

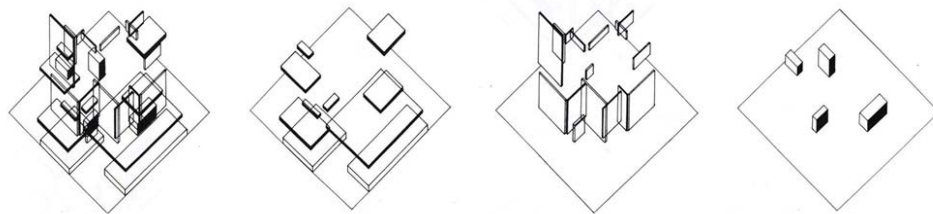


Figure 2-38: Doesburg's counter-construction from Eesteren project, 1923

2.6. Languages and configurations

One key idea of the constructive systems is that the information given by the input may be considered as a list of instructions to be followed or as a list of data to be acted upon. In the first case the rules provide all the necessary information to construct the object; in the second case, the data have to be acted upon by a schema encoded within the system. The first case may be referred as a 'language' and the second as a 'configuration' (Stiny and March 1981)

This last part of this exposition of formal methods will deal primarily with the configurations aspect of the constructive systems. A central point in the 'Geometry of Environment' (March and Steadman 1971) is that design is a mode of computation that explicitly exercises both imagination and reason. In this sense a formal theory of spatial design is directly linked to Alberti's (1486) worldview and provides a syntactic procedure for creation of designs based on speculative knowledge. The configurations themselves, that is the schemata that particularize the data, can be modeled after syntactic and semantic domains; the syntactic domains are typically given in terms of arithmetic and geometry respectively; the semantics under functional ones. These three categories of dealing with configurations can be nicely mapped to the Froebel categories of form: a) forms of knowledge to which are related quantities of modern proportion and spatial relations; b) forms of beauty to which are attributed qualities of spatial transformations and symmetries; and c) forms of life to which are ordered and configured the spatial representations of actual objects with functional semantic meaning attached to them (March 1992). This mapping can be furthered to relate well-known existing categories of formal inquiry in architecture design such as proportion, symmetry and compartment with arithmetical, geometrical and semantic elements and relationships and this is in fact the mapping given in the area of studies known as 'architectonics' (March 1998), (Economou 1998). A brief exposition of the first two domains is given here, proportion and symmetry, to prepare the ground for the formal exposition of the model introduced in this thesis.

2.6.1. Proportion

Ratio and proportion have been significant concepts in architectural design since the first surviving treatise in architecture design by Vitruvius (Morgan 1914) and still were dominant

fields of inquiry throughout the twentieth and twenty-first century. Ratio is a relation between two numbers and a proportion is a relation between two ratios. The last number that can establish a proportion is three. For three numbers x , y and z and $x < y < z$ there are three possible outcomes of comparisons, one unique case of equality $x:y=y:z$ and two cases of inequality $x:y < y:z$ and $x:y > y:z$. For each case of inequality, there can be an infinite number of subcases with respect to the actual value of the number involved in the comparison. Among these relationships some are more interesting than others. For example for three numbers x , y and z and $x < y < z$ if $(1/z) - (1/y) = (1/y) - (1/x)$ the inequality can be written as an equality, namely, $(z-y)/z = (y-x)/x$. The problem has been nicely solved in antiquity by Greek mathematicians in a series of successive attempts initially proposing two more such equalities by Archytas, the arithmetic and the harmonic ones, later three more, possibly by Eudoxus, and finally two additional distinct sets of four by Nichomachus and Pappus respectively, with three overlapping cases between them, bringing the total number of equalities to ten. These ten relationships of ratios plus the initial one of equality, the geometric mean, brought the number of comparisons to eleven and they are all treated informally under the heading of proportionality theory or theory of means (Heath, 1921).

Figure 2-39 shows three out of eleven ways of comparing two ratios involving three magnitudes, namely the arithmetic, geometric and harmonic ratios (March 1998).

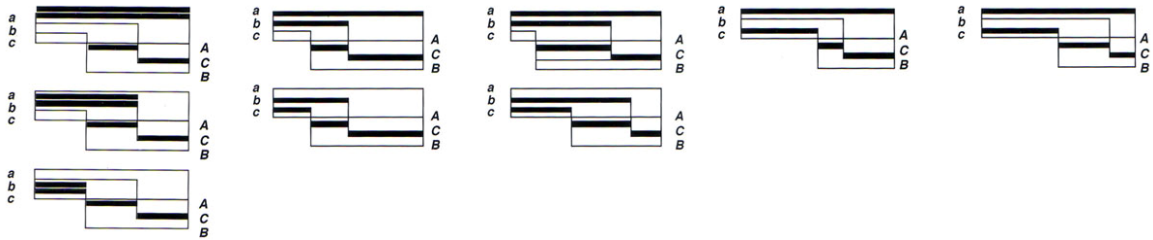


Figure 2-39: Means a) arithmetic; b) geometric; c) harmonic

All root ratios can be nicely depicted by a shape grammar of adding successively squares starting with an initial shape of a square (March 1998). If the derivation of the grammar is shown in a tree, several sets of ratios with spatial properties are depicted including in the extremes the unit ratios such as 2:1, 3:1, 4:1, 5:1 to the left, and the Fibonacci ratios 2:1, 3:2, 5:3, 8:5 to the right as well as all the Nichomachean ratios such as the multiplex, superparticular, superpartiens, multiplex superparticular, and multiplex superpartiens (Heath 1921). The initial derivation of the grammar is shown in Figure 2-40.

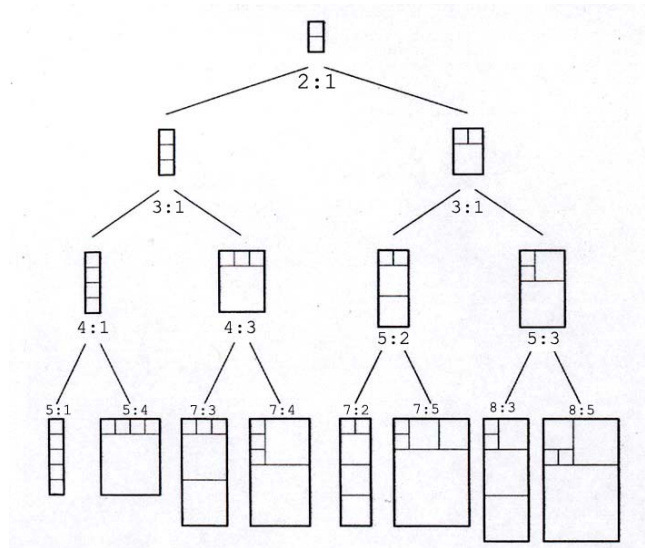


Figure 2-40: Configuration and proportion: Schematic derivation of root ratios

Ratios and proportion provide a very rich vocabulary for systematic studies in formal composition and March and several others have offered a considerable body of work foregrounding arithmetical relationships in form analysis and synthesis. Among these case studies the Schindler's system for the How House has been prominently featured as a case study in twentieth century architecture that exemplifies principles of classical composition. Figure 2-41 shows few orthographic projections of the R.M Schindler How House (March 1993).

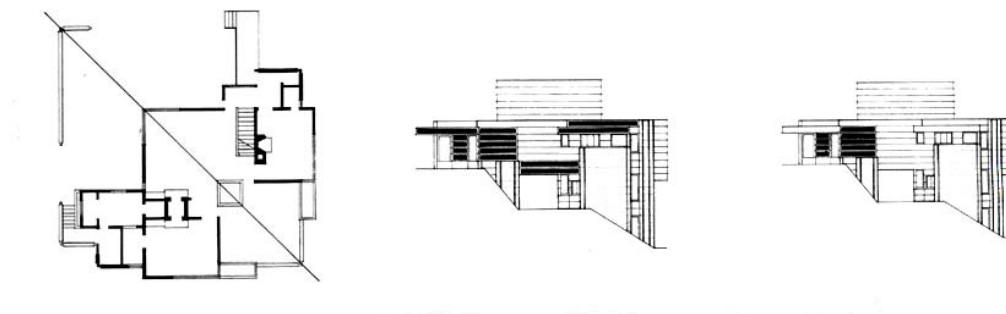


Figure 2-41: R.M Schindler's How house

2.6.2. Symmetry

'If I am not mistaken, the word symmetry is used in our everyday language with two meanings. In the one sense symmetric means something like well-proportioned, and well-balanced, and symmetry denotes that sort of coincidence of several parts by which they integrate into a whole. Beauty is bound up with symmetry... The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times: bilateral symmetry, the symmetry of left and right, which is so conspicuous in the structure of the higher animals, especially the human body. Now this bilateral symmetry is strictly geometric and, in contrast to the vague notion of symmetry discussed before, an absolutely precise concept.' Weyl (1952)

Symmetry as one understands it today is radically different from the Greek word 'symmetria' from which it derives. A nuance of the old meaning of 'symmetria' still survives in the new context and is nicely illustrated in the above quotation by Hermann Weyl (1952) from his classic book on symmetry. To the Greeks 'symmetria' (συμμετρία < συν+μετρον: with measure) meant commensurability and it was suggested to be a canon of beauty in nature and in art. Two magnitudes are said to be commensurable if there exists a third magnitude that divides them both without remainder. As applied to works of art, symmetry meant commensurability of the parts of a work to one another and to the whole; in other words, a work of art was considered symmetrical if all the parts were exact multiples of a visible part of this work, a module. A rather blurred account of this notion of symmetry was given by the Roman architect Vitruvius in his treatise 'De Architectura'.

"Symmetry is a proper agreement between the members of the work itself, and relation between the different parts and the whole general scheme, in accordance with a certain part selected as standard...In the case of temples, symmetry may be calculated from the thickness of a column, from a triglyph, or even from a module; in the ballista, from the hole or from what the Greeks call the περιτρητος; in a ship, from the space between the tholepins (διαπηγμα); and in other things, from various members"...

... "The design of a temple depends on symmetry, the principles of which must be carefully observed by the architect. They are due to proportion, in Greek αναλογια.

Proportion is the correspondence among the measures of the members of an entire work, and of the whole to a certain part selected as standard. From this result the principles of symmetry". (Vitruvius, [c.50 BC] Morgan, 1914, 14 and 72)

Both definitions quoted above illustrate Vitruvius' attempt to give an appropriate definition of symmetry and its relation to proportion; the first passage refers to the fundamental principles of architecture and the second passage refers on principles involved in the design of temples. Symmetry was asserted to be the key to perfection; the canons of antiquity were attempts to capture this idealized beauty by imposing an order and a rationale in their construction. This intellectual conception of beauty as guaranteed by symmetry and expressed in the Pythagorean and Platonic doctrines defined a line of thought which still pervades the formulation of various compositional techniques in architecture and music.

Another conception of symmetry came in the foreground in Renaissance with the doctrine of the Golden Section, or as it was then generally known, the 'Divine Proportion' or 'Sectio Aurea' (Pacioli 1506). In this context, symmetry results from the systematic application of a single proportion; this proportion is essentially the proportion between two relations that have one term in common and one of the three terms is the sum of the other two terms. This proportion arises when a line is divided in extreme and mean ratio, that is, in such a way so that the ratio of the whole line to the greater line is equal to the ratio of the greater line to the smaller line. This symmetry, based on the extreme and mean ratio, was asserted to be the universal key to perfection in nature and in art; it sacrificed exact commensurability but it imposed a single ratio throughout. According to this definition, an object was assumed to be symmetrical when all its parts were related to one another and to the whole not by means of commensurable ratios as in 'symmetria', but by means of one single incommensurable ratio, namely the ϕ . Inherent in this conception of symmetry were the beliefs that the same principles of perfection apply in nature and in art with the additional doctrine that the ideal symmetry was conditioned mathematically by the Divine Proportion. This kind of symmetry was revived in the middle of the nineteenth century by A. Zeising (1854) and since then it comes and goes in the foreground of the scientific and artistic milieu by means of writings and works of researchers and artists. However, these works and books, even if they have been influential they are not convincing as to the universal and ubiquitous role of the extreme and mean ratio.

The modern conception of symmetry developed around the notion of repetition and is a strictly geometric and precise concept. The origins of the modern conception of symmetry and the shift

from the concept of 'symmetria' are to be found in the writings of the Florentine architect and theoretician Alberti and especially, in his treatise on the art of building, 'De re Aedificatoria':

"Look at Nature's own works...right should match left exactly. We must therefore take great care to ensure that even the minutest elements are so arranged in their level, alignment, number, shape, and appearance, that right matches left, top matches bottom, adjacent matches adjacent, and equal matches equal... I have long been an admirer of the ancients in which they displayed outstanding skill: with statues, especially for the pediments of their temples, they took care to ensure that those on the one side differed not a whit, either in their lineaments or in their materials, from those opposite". (Alberti, [1486] Rykwert et al, 1988, 310).

Bilateral symmetry, that is, the identical disposition of a theme or a motif about both sides of an imaginary axis, is just one of several types of symmetrical configurations that are composed by identical parts. In this case, the bilateral symmetry of the design is induced by a reflection about a mirror line or a mirror plane passing through the middle point of the design. Other transformations that induce symmetrical designs are rotations, inversions, translations, glide reflections, screw rotations and others. Bilateral symmetry, that is, the identical disposition of a theme or a motif about both sides of an imaginary axis, is just one of several types of symmetrical configurations that are composed by identical parts. The incorporation of these isometries in the study of symmetry occurred gradually. The concept of symmetry has undergone substantial changes over the time. Originally, and very much in the same line of thought with Alberti's principles, the symmetries of shapes were related exclusively with mirror reflections in planes. Simple rotation axes were added later to the symmetry planes to construct the symmetry classes of the finite figures. The transformations of translation, screw rotation, glide reflection and rotor reflection were introduced to construct the symmetry classes of infinite figures. Infinite small translation and rotations were added to construct limiting symmetry classes. The quest for finding a single principle for the construction of any symmetrical figure was initially resolved by Wulff in 1897 and Viola in 1904 in their proof that all symmetry transformations of finite figures in three-dimensional space are reduced to successive reflections in no more than three planes, which might not be symmetry planes of the figure. This single principle was finally established by Boldyrev in 1907 in his proof that all symmetry transformations of finite and infinite figures in three-dimensional space are reduced to successive reflections in no more than four planes which may not be symmetry planes of the figure. Figure 2-42 shows some architectural and urban

design projects whose configurations are described by specific symmetry structures. These projects include Schindler's Lowes House with one reflectional axis, Sir Sloane's Sepulchral Church with three reflectional axes, and Kahn's Hurva Synagogue for with four reflectional axes, Meier's Karlsbad Apts, with a translational structure and Goff's Price Studio with nested rotational and reflectional axes of different order. A complete presentation of the formal language to discuss symmetry will be given in the next chapter.

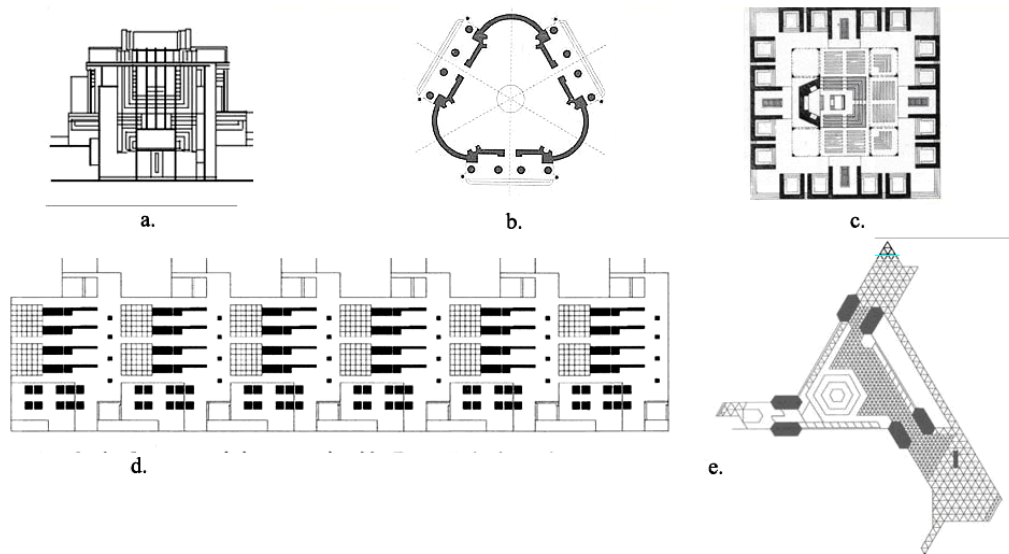


Figure 2-42: Configuration and symmetry. A) Schindler; b) Sir Soane; c) Kahn; d) Meier; e) Goff

The study of configuration and the ways it informs architectural composition is an important aspect of formal analysis and synthesis in architecture composition. One of the biggest challenges that this study faces is its ability to participate actively in a variety of design contexts and not only when the presence of arithmetical or symmetrical configurations is the prevalent design solution. There is no doubt for example that any design in the Beaux-Arts tradition has to exhibit specific orders of symmetry and proportion but the challenge is to show how a better understanding of this body of knowledge – both in terms of the configurational possibilities as well as the mathematical language that describe them – can be used to any design context, even those that do not necessarily evoke such as a formal or programmatic constraint. To this prospect one will turn next after the presentation in the next chapter of the fundamentals of group theory, the mathematical language of configuration and pattern.

2.7. Summary

An overview of formal systems in architectural design has been given and their role in the systematic description, interpretation, and evaluation of existing works of architecture as well the systematic creation of new works of architecture has been described in depth. The overview here used a basic algorithmic structure for the foundation of formal systems (Stiny and Gips 1978) and all various examples of systems were presented within this framework. Two types of formal systems were reviewed, the analysis system and the design system and they were both presented as frameworks that utilize constructive and evocative modes of interpretation. The chapter concluded with an informal presentation of applications in formal analysis and design based on group theory as it pertains to the study of proportion and symmetry in architecture.

Chapter 3 Symmetry and Group Theory

Chapter 3 deals with the fundamentals of the formal system used in this research, group theory. All basic constructs used in the research and based in group theory are presented here: group definitions, pictorial and discursive representations, graph representations, Cayley diagrams, group classifications, partial order lattices, isomorphisms, automorphisms, as well as permutations and combinatorics. All representations are given with respect to a singular structure, the symmetry group of the square, the dihedral group of order eight, to illustrate systematically the diverse aspects of the structure that each representation foregrounds.

3.1. Introduction

'The theory of groups is, as it were, the whole of mathematics stripped of its matter and reduced to pure form.' Poincare (1905)

'Numbers measure size; groups measure symmetry.' This first sentence of Armstrong's textbook 'Groups and Symmetry' (1988) is striking. Indeed the applications and the insights that group theory offers are many. From its first appearance disguised in the theory of equations to describe the effect of mapping of the different roots of a polynomial equation into themselves, to its various applications to number theory, combinatorics and especially to symmetry theory of geometrical figures there are many fascinating applications to explore. The focus here is the exploration of the application of group theory in symmetry theory. This viewpoint requires a formulation of a mathematical characterization of symmetry. And still, the formalization is not enough; Klee asserts that the bilateral conformity of two parts, that is, the old definition of symmetry, has been superseded by the equalization of unequal but equivalent parts (Klee 1953). For Klee, the purely material balance of the scale finds its counter-part in the purely psychological balance of light and dark, weightless and heavy colors. Klee is right: It is the balancing and proportioning power of eye and brain that regulates the characterization of the object in terms of equilibrium and harmony. But all such entire world-making requires foundations. It is the premise of this work that all studies in formal composition should start from foundations and expand upon them. Group theory is a part of this foundation and it is argued here

that it is a powerful tool that allows for possible re-descriptions in the analysis and description of an architecture work.

Here a very brief account of the history and logic of group theory is given to provide the foundations for the development of the model developed in this work. Formal accounts of group theory and in-depth analyses of its applications in the arts and particularly in the visual arts and architecture have been given in various sources and several of them are mentioned in this work below. The mathematical study of transformations, symmetry groups and abstract groups in general, has been given in various sources (Armstrong 1988); (Baglivo and Graver 1983); (Budden 1972); (Coxeter 1969); (Coxeter and Moser 1972); (Dorwart 1966); (Grossman and Magnus 1964); (Grunbaum and Shepard 1987); (Jeger 1966); (Lockwood and MacMillan 1978); (March and Steadman 1971); (Maxwell 1975); (Shubnikov and Koptsik 1974); (Toth 1964); Yaglom (1962); (Yale 1968); (Weyl 1952). The emphasis here is given to the representations and ways that group theory can be used to explain complexity in architectural design analysis and synthesis. A brief exposition of the formalism along with the formal tools that are used in the analysis and synthesis of form is given in the first part and a brief historical survey of the advance of group theory completes the chapter.

3.2. A first encounter

'The integers of shapes form a group, the underlying logical structure of math can be exposed in an aesthetic and satisfying manner.' Cauchy (1840)

'A geometry is the study of the properties of a set S which remain invariant when the elements of set S are subjected to the transformations of some transformation group.' Klein (1921)

Group theory is the mathematical language of symmetry. There is no better way to understand the foundations and the premises of group theory than within one of its major applications in the study of geometrical figures. Let's take a square embedded in the plane E denoted by four vertices A, B, C, D. Two axes R_1 and R_2 are drawn perpendicular to the mid-edges of the square,

and two more axes R_3 and R_4 intersect at the point O , the center of the square. The square $ABCD$ with the four axes and its center O are shown in Figure 3-1.

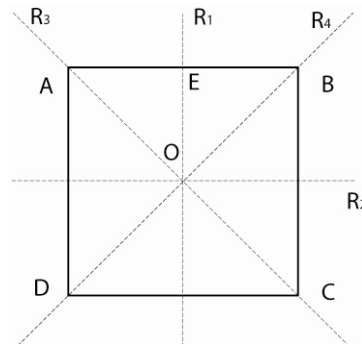


Figure 3-1: A square ABCD

Among all possible transformations of the plane E that contains the shape $ABCD$, there are eight that are quite special for the structure of the square $ABCD$. These transformations are: a reflection r_1 in the line R_1 bisecting the edges AB and CD ; a reflection r_2 in the line R_2 bisecting the edges BC and DA ; a reflection r_3 in the leading diagonal line R_3 connecting A and D ; a reflection r_4 in the secondary diagonal line R_4 ; a rotation s_{90} by 90° clockwise about the center O of the square; a rotation s_{180} by 180° clockwise about O ; a rotation s_{270} by 270° clockwise about O ; and a rotation s_0 by 0° or 360° clockwise about O , or more generally, a “do nothing” transformation, typically denoted by the symbol e . All these transformations are quite special in the way that when they operate on the square even if they move the individual vertices and edges of the square from one position to another, the overall shape appears unchanged. For example, in the case of the square $ABCD$, the transformation s_{90} puts A in the position occupied by B , B in the position occupied by C , C in the position occupied by D , and D in the position occupied by A . The transformation r_4 leaves A and C where they are and interchanges the points B and D respectively. Each of these eight transformations is called ‘symmetry of $ABCD$ ’. The collection of all these symmetry transformations that leave the structure of the square invariant is given in set notation below.

$$D_4 = \{e, s_{90}, s_{180}, s_{270}, r_1, r_2, r_3, r_4\}$$

An alternate representation of these transformations is suggested by the mapping of shapes upon these transformations. In this manner the eight transformations that leave the shape invariant may be represented as eight shapes that all together form the visual structure of the square. The square

then is seen as an aggregation of eight lines that taken together form the structure of the square. The original line $AB/2$ is taken as the identity and all other parts are derived by the application of the transformations above. The collection of all these parts that form a square is given in set notation in

Figure 3-2. All segments correspond one to one to the set of transformations above.

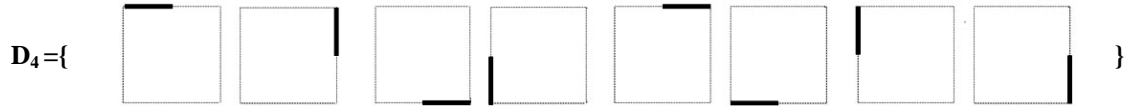


Figure 3-2: The eight parts of the square in set notation

This collection of transformations has remarkable properties. To begin with, they can all be combined using a rule as simple as the injunction “followed by”. In this case, transformations are combined in series of any desired length to denote sequences of transformations the one following the other. There are alternative representational schemes that capture the conventions of such rules applications; here the symbol $(*)$ is used to denote the rule “followed by” and the sequence of operations is meant to be read from right to left. For example a transformation $x*y$ means that a transformation y is followed by the transformation x . In the example of the square ABCD, a rule sequence such as r_1*s_{180} means that the rotation s_{180} by 180^0 clockwise about O is followed by the reflection r_1 in the line R_1 . In this case the combined transformation r_1*s_{180} interchanges the points A and D as well as B and C. The result of the combined transformation is the same as the reflection r_2 in the line R_2 . In this case one could write $r_1*s_{180} = r_2$. Any sequence of transformations that is produced by the combination of the eight transformations of the square will always produce one of the eight original transformations of the square. This is a property of this system that is formally called ‘closure’, that is, for any two elements x, y belonging in a set, their products, and in this case, the products $x*y$ and $y*x$, also belong in the group. But this collection of eight transformations has more exciting features that all nicely illustrate fundamental aspects of the theory of algebraic structures and more specifically of group theory.

First, there is a transformation that basically does nothing, the so-called identity transformation. This transformation in the example of Figure 3-1 is the transformation e . It is also clear that for each transformation in the set is another operation that can cancel it. In other words this means

that for each transformation there is one that when it is combined with it produces the identity transformation. For example all mirror transformations when they are applied twice, bring the shape to its original position. Similarly two rotations by 180^0 clockwise about O bring the shape back to itself. Slightly more interesting but nevertheless obeying the same case are the rotations by 90^0 and 270^0 clockwise about O. It is easily seen that the rotation by 90^0 clockwise about O is the inverse of the rotation by 90^0 clockwise about O and the rotation by 270^0 is the inverse of the rotation by 90^0 , both respectively clockwise about O. The identity is a bit more abstract in this but is essentially the same; the identity can be followed by itself and will have the shape stay stationery in its original position. This property of the system is the so-called *inverse* property. Finally, it is clear that for any three operations that are constructed as a series of the one following the other, the transformation $(x*y)*z$ is equivalent to doing z then y and x , as is doing the sequence z and y and then x , a sequence denoted as $x*(y*z)$. All these are remarkable properties that are all captured in the formal definitions of group theory. A succinct account follows below.

3.3. Group structure

‘Geometry is symmetry – It was Felix Klein who made symmetries fundamental and geometries subsidiary. The essence of Klein’s program is that geometry is group theory’ (Klein, [1872] Stewart et al, (1992)

A group is a set endowed with a rule; the set can be any collection and the elements of the set are whatever comprises this collection. The rule combines any ordered pair x, y of elements of the set and obtain a unique product xy which also lies in the set; from this definition it follows that both possible ways of combining any two elements, x, y , that is, xy and yx , also lie in the group. The rule is usually referred to as a multiplication or a composition on the given set.

A group is a set G together with a rule on G which satisfies three axioms: a) the multiplication is associative, that is to say, $(xy)z = x(yz)$ for any three, not necessarily distinct elements in G ; b) there is an element e in G , called an identity element, such that $xe = x = ex$ for every $x \in G$; c) each element in G has an inverse x^{-1} which belongs to the set G and satisfies $x^{-1}x = e = xx^{-1}$. In

general $xy \neq yx$ (*is different*). However, when certain pairs of elements x, y in G obey $xy = yx$, it is said that these elements commute. The identity element e commutes with all elements of a group and every element commutes with its inverse. If all elements in G commute with each other, i.e., $xy = yx$ for all x, y of G , the group G is called commutative or Abelian.

A group is abstract if its elements are abstract, i.e., if they are not defined in any concrete way. A concrete example of an abstract group, i.e. a group with concrete elements with a law of composition, is called a realization of that abstract group. Such realizations might be groups of numbers, matrices, or geometric transformations. The structure of a group is the statement of the results of all possible compositions of pairs of elements. In general, the structure of a group can be defined in an analytical and a constructive way. The information about the structure of a group can be encoded descriptively in the explicit enumeration of all possible combinations of pairs of elements or constructively by a set of few elements and a set of rules that determine all possible combinations of pairs within the group. The analytical description of the structure of the group is given usually in a square array, the so-called multiplication table of the group, and the constructive description is given in a set of group generators and defining relations that apply on the generators.

3.3.1. *Multiplication table*

The explicit description of the structure of the group is given usually in a square array, the multiplication table of the group. This array provides the results of all combinations of the elements of the group. The multiplication table consists of columns and rows containing each a rearrangement of the elements of the group, and each entry in the square array denotes the product of the combination of the elements positioned in the corresponding outer rows and columns. This way of representation of the structure of the group was introduced in 1854 by Cayley and is very similar to the familiar multiplication tables of arithmetic. It follows from the definition of the multiplication table that for a finite group of order n , all possible binary combinations of the n elements are given by the formula n^2 . These n^2 products are explicitly given in the multiplication table of the group. The concept of the multiplication table is illustrated here in Table 1 the explicit illustration of the structure of the symmetry group of the square. The product $x*y$ means first perform y then x ; the products 8^2 of all four groups are computed by first taking the symmetry element in the top row and then combining it with the corresponding element in the left column.

Table 1. The multiplication table for the structure of the square

	e	s_{90}	s_{180}	s_{270}	r_1	r_2	r_3	r_4
e	e	s_{90}	s_{180}	s_{270}	r_1	r_2	r_3	r_4
s_{90}	s_{90}	s_{180}	s_{270}	e	r_4	r_1	r_2	r_3
s_{180}	s_{180}	s_{270}	e	s_{90}	r_3	r_4	r_1	r_2
s_{270}	s_{270}	e	s_{90}	s_{180}	r_2	r_3	r_4	r_1
r_1	r_1	r_2	r_3	r_4	e	s_{90}	s_{180}	s_{270}
r_2	r_2	r_3	r_4	r_1	s_{270}	e	s_{90}	s_{180}
r_3	r_3	r_4	r_1	r_2	s_{180}	s_{270}	e	s_{90}
r_4	r_4	r_1	r_2	r_3	s_{90}	s_{180}	s_{270}	e

All the properties of the group structure discussed so far are seen in a glance in this representation. More specifically, the representation of the groups in terms of the multiplication tables has three properties: a) each row and column contains each symbol exactly once; this property of the array corresponds to a fundamental theorem connecting groups and permutations; b) there is one row and one column that are identical with the top row and left column of the array respectively; this property of the array corresponds to the group axiom of the unit element or identity; c) the two entries e for the intersections of two elements in the array are symmetrically located with respect to the main diagonal; this property of the array corresponds to the group axiom on the existence of inverses.

3.3.2. Group generators

The constructive representation of a group describes groups independent of order and is given in a set of group generators and defining relations that apply on the generators. For an element x in a group G , by the axiom of closure, it follows that all powers of x , that is, x , xx , xxx , ... or otherwise, x^1 , x^2 , x^3 , ... all belong in the group. Furthermore by the axiom of inverses the elements x^{-1} , $x^{-1}x^{-1}$, $x^{-1}x^{-1}x^{-1}$, ... or otherwise x^{-1} , x^{-2} , x^{-3} , x^{-4} , ... all belong in the group too. x^0 is defined to be $x^0 = e$. If all elements of the group can be expressed as products involving only one element x and its inverse x^{-1} , then this element x is called a ‘generator of this group’ and the group

is called a ‘cyclic group’. If $x^n = e$ for some $n > 0$, the least such n is called the ‘order of x ’ and the element x is said to have a ‘finite order’. If no such n exists, x is said to have ‘infinite order’.

Similarly for two elements x and y in a group G then by the axiom of inverses, x and y are also in the group and so are $x^{-1}yx$, $xyx^{-1}y$, and so on. Any product that can be written using x and y as factors in any sequence and with any finite frequency is an element of the group and is called a ‘word’ (Baglivo and Graver, 1976). If all elements of the group can be expressed as products involving the elements x and y and their inverses x^{-1} and y^{-1} , then x and y are called the ‘generators of the group’ and the corresponding group is called a ‘dihedral group’. The concept of group generators can be extended to a set of more than two elements. If S is a set of elements of a group G and all elements of G can be expressed as products involving only the elements of S and their inverses, then the elements of S are the generators of G . Still, the generators are not enough to build by themselves the characteristics of the group; what is needed is an explicit definition of their relationships one with another. The second ingredient for the complete description of the structure of the group is a set of rules, that is, a set of defining relations that determine the structure of the group by group relations. If w is a non-empty word of group G such that $w = e$, then this equality is a relation of G . Since the word w is a product of generators of G , $w = e$ is a generating relation of G . The word w can be any product of generators and their inverses in any sequence and with any finite frequency. Two words w_1, w_2 that represent the same group element are equivalent words in a group G ; equivalent words are in the same class and any of those can be taken as a representative of the class. The concepts of group generators and defining relations are used here to illustrate the symmetry structure of the square. The generators for the symmetry group of the square are the rotation of 90° about the center s and a reflection r about the mid-edge of the square. The defining relations are given by forming the words $w = e$ for each element separately and for the two together. The concept of group generators and defining relations can be easily generalized. A more intuitive description of these defining relations is given in the next section on the pictorial representation of the graph of the group. The relations for s, r and $s*r$ are given in Table 2.

Table 2: Generators and defining relations for the symmetry group of the square

Generators	s		Rotation of 90° about the center
	r		Reflection about an axis passing through the mid-edges
Relations	s^4	=	e
	r^2	=	e
	$rsrs$	=	e

3.3.3. *Pictorial representation*

A group can be defined as a network of directed segments specifying within its structure how any product of group elements corresponds to successive paths on the graph network. The representation of a group as a network of directed segments where the vertices correspond to elements and the segments to multiplication by group generators and their inverses, was invented by Cayley as well. Such a network or graph is often called a ‘Cayley diagram’.

All graph networks of groups have in common certain fundamental properties (Baglivo and Graver 1983). a) Every vertex of the graph is in one-to-one correspondence with a group element; b) Every edge of the graph is a directed segment and all edges with the same weight correspond to a ‘singular group generator’; c) Every path or sequence of directed segments within the graph corresponds to a word that represents a group element, and vice versa; d) Every succession of two paths within the graph corresponds to a composition or multiplication of two group elements; e) Any word for e corresponds to a closed path on the graph; and f) The graph of a group is a ‘connected network’; that is, there are paths from each vertex to every other vertex.

The Cayley diagrams are extremely useful tools in the examination of the structure of symmetry groups. The use of few rules, the so-called generators of the group, produce visually all the words that correspond to the vertices of the graph and characterize automatically the defining relations that control the structure of the groups; the complexity of the multiplicity of words is easily resolved within the structure of the graph. Significantly, all closed paths generate the defining relations for the group; closed paths involving a minimum number of steps or nodes are the defining relations mostly used. The sets of transformations that bring the shape into coincidence with itself can be nicely visualized as sets of directed segments or arrows that move or transform parts of the shape while the overall shape remains invariant. The resulting graphs can be interpreted in perspectival projections, that is to say, seen as if the observer looks at the shapes through their front face in perspective so that the frontal face is closer and bigger and the back face is smaller and its edges are parallel to the front edges. The concept of the Cayley diagram is used to illustrate the symmetry structure of the square in Figure 3-3.

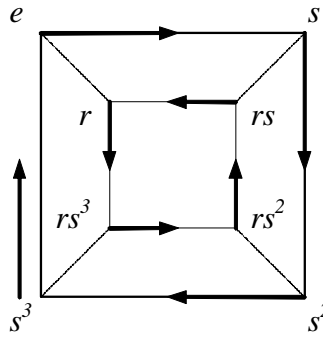


Figure 3-3: The Cayley diagram of the symmetry group of the square

3.3.4. Subgroups

One of the most interesting aspects of the structure of the group lies in its construal of its part (\leq) relation. The elements of a symmetry group with an order n can form 2^n subsets. For example, in the case of the square there are $2^8 = 256$ possible subsets of symmetry transformations that leave the structure of the square invariant. Still very few of those have the very same properties with the symmetry group of the square that are part of. In general the subsets of the groups that form smaller groups within the big one are called subgroups and one of the most fascinating aspects of group theory is that the order of a finite group and the order of any of its subgroups are numerically related. This assertion is due to Lagrange back in 1771. Lagrange's theorem states that if H is a subgroup of a group G , and if the order of G is n , then the order m of H is a factor of n . In other words, the theorem specifies that the order of a finite group is a multiple of the order of any subgroup. From this, it follows that all prime-order groups have no proper subgroups.

There are several techniques that are used in the identification and generation of subgroups. In general, the enumeration of all subgroups for a given finite group is a very difficult task and has been carried through only for few groups; however, the generation of subgroups for groups of a small order is straight forward. The cyclic subgroups can be immediately picked out because every element of a group may be used to generate a cyclic subgroup. Given a group G and an element x of G , the set of all powers of x is a subgroup of G . This subgroup is called the subgroup generated by x and is written as $\langle x \rangle$. If x has finite order m , then $\langle x \rangle = \{e, x^1, x^2, x^3, \dots, x^{m-1}\}$. If x has infinite order, then $\langle x \rangle$ consists of infinite elements. In both cases the order of x is precisely the order of the subgroup generated by x . If there is an element x in G such that $\langle x \rangle = G$, then G is a cyclic group.

Similarly, any subset of a group may be used to generate a subgroup. Given a group G and two elements x, y in a subset H of G , the set of all powers of x and y and their combinations is a subgroup of G . An expression of the form $x^m y^n$ for m, n any integers is a word in the elements of H . The collection of all these words is a subgroup of G . This subgroup is called the subgroup generated by H and written $\langle H \rangle$. If there are elements x, y in H such that $\langle H \rangle = G$, then the set H is a set for generators for G . The idea of a group generated by one or two elements may be extended for any number of generators.

Still, not every subset H is a subgroup. For example the subset $H = \{s_{90}, r_1, r_3\}$ fails because $s_{90} r_1 = r_4$, an element that does not belong in the set. The same set fails for many more reasons too, perhaps the most obvious being that it does not contain the identity. In all cases, for h and k lie in H then it should be: a) $h*k \in H$; b) $h^{-1} \in H$; and c) $I = h*h^{-1} \in H$. Table 3 displays the complete catalogue of the subgroups identified within the structure of the square. The symmetry group of the square has in all ten different subgroups.

Table 3: The ten subgroups of the symmetry group of the square

<i>Subgroups</i>	<i>Order of symmetry</i>
$\{e, s_{90}, s_{180}, s_{270}, r_1, r_2, r_3, r_4\}$	8
$\{e, s_{180}, r_1, r_2\}$	4
$\{e, s_{180}, r_3, r_4\}$	4
$\{e, s_{90}, s_{180}, s_{270}\}$	4
$\{e, r_1\}$	2
$\{e, r_2\}$	2
$\{e, r_3\}$	2
$\{e, r_4\}$	2
$\{e, s_{180}\}$	2
$\{e\}$	1

For a group G , a subgroup H of G , and an element $x \in G$, the set of elements $x*H$ defined by $x*H = \{x*h: h \in H\}$ is called a ‘left coset’ of H in G . The set of elements $H*x$ defined by $H*x = \{h*x: h \in H\}$ is called a ‘right coset’ of H in G . $x*H$ is the left coset of H containing (or generated by) x . $H*x$ is the right coset of H containing (or generated by) x . This rather abstract notion of the left and the right coset of a group can be nicely exemplified within the structure of the square. For the

symmetry group of the square $D_4 = \{e, s_{90}, s_{180}, s_{270}, r_1, r_2, r_3, r_4\}$ and a subgroup $J = \{e, r_1\}$, for any element $x \in D_4$ the coset $J*x$ is defined to be:

$$\{e*x, r_1*x\}$$

The coset is formed by multiplying each element of J by x and collecting the resulting elements into single sets. These sets are given in Table 4.

Table 4: The left cosets of the subgroup $J = \{e, r_1\}$ for the symmetry group of the square

$J*e$	=	$\{e*e, r_1*e\}$	=	$\{e, r_1\}$
$J*s_{90}$	=	$\{e*s_{90}, r_1*s_{90}\}$	=	$\{s_{90}, r_4\}$
$J*s_{180}$	=	$\{e*s_{180}, r_1*s_{180}\}$	=	$\{s_{180}, r_2\}$
$J*s_{270}$	=	$\{e*s_{270}, r_1*s_{270}\}$	=	$\{s_{270}, r_3\}$
$J*r_1$	=	$\{e*r_1, r_1*r_1\}$	=	$\{r_1, e\}$
$J*r_2$	=	$\{e*r_2, r_1*r_2\}$	=	$\{r_2, s_{180}\}$
$J*r_3$	=	$\{e*r_3, r_1*r_3\}$	=	$\{r_3, s_{270}\}$
$J*r_4$	=	$\{e*r_4, r_1*r_4\}$	=	$\{r_4, s_{90}\}$

Several observations can be extracted from the computation in Table 4: a) the first four sets are equal to the last four sets forming thus only four distinct cosets; b) one of the four cosets is the subgroup J itself; c) no distinct cosets have any element in common; d) every element in D_4 lies in some coset; and e) each coset has the same number of elements. It is clear from observations (b) and (e) that each coset has two elements; from (a) and (c) it is deduced that the cosets when taken all together produce $4*2 = 8$ elements which nicely explains that the order of J divides that of D_4 and that the result of doing the division will be the number of cosets.

Several other corollaries of these observations can be generalized and proved. These corollaries include: a) for a subgroup H of a group G , the G is a disjoint union of left cosets (or alternatively right cosets) of H in G ; b) for a finite group G the order of an element of G divides the order of G ; and c) every group of prime order is cyclic. All such observations are derived from the precise numerical relationship between groups and subgroups and the core of the Lagrange theorem. Here for example, using this theorem one sees that if there are any subgroups in the symmetry group of

the square that has an order of symmetry eight, these subgroups cannot have any orders other than 1, 2, 4, and 8. Still, this does not guarantee that these do exist. The only theorem for that is Sylow's theorem that proposes that if a number m is a power of a prime k and divides the order of a group n , then the group has a subgroup of order m . For the structure of the D_4 the possible orders of symmetry subgroups are 1, 2, 4, and 8. The numbers 2 and 4 are powers of a prime 2 and therefore there should be subgroups with such order. The ten subgroups of the symmetry group of the square are pictorially illustrated in Figure 3-4. Here each transformation that leaves the square invariant is mapped as a line equal to AE equal to the half length of the side AB of the original square.

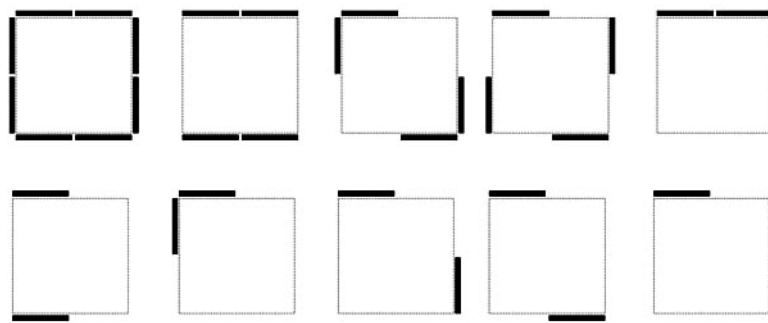


Figure 3-4: The ten subgroups of the symmetry group of the square

The same subgroup relation can be used to structure all possible symmetry subgroups of space. In fact all symmetry groups are subgroups of the Euclidean group G that consists of all possible isometries in the Euclidean space. All these symmetry groups are classified according to their translational structure and the dimensionality of the space that contains their elements (Yale, 1964). In Euclidean space there are ten symmetry groups G_{ij} , for i = number of axes of translation and j = dimension of space, and $i \leq j$ and $j \leq 3$ as given in Table 5 (Economou 1999).

Table 5: The ten group structures of Euclidean space

	<i>0-dimensional</i>	<i>1-dimensional</i>	<i>2-dimensional</i>	<i>3-dimensional</i>
	<i>groups</i>	<i>groups</i>	<i>groups</i>	<i>groups</i>
<i>Point groups</i>	G_{00}	G_{01}	G_{02}	G_{03}
<i>Line groups</i>		G_{11}	G_{12}	G_{13}
<i>Plane groups</i>			G_{22}	G_{23}
<i>Space groups</i>				G_{33}

The ten symmetry subgroups G_{ij} of the Euclidean space, for i the number of axes of translational symmetry and j the dimension of space, can further be decomposed in a series of subgroups according to the types of symmetry elements they contain and the ways these elements interact. There is one zero-dimensional point group G_{00} , one one-dimensional point group G_{01} , two one-dimensional line groups G_{11} , two two-dimensional point groups G_{02} , seven two-dimensional line groups G_{12} , seventeen two-dimensional plane groups G_{22} , fourteen three-dimensional point groups G_{03} which further split to seven finite polyhedral ones and seven infinite prismatic groups, nineteen three-dimensional line groups G_{13} , eighty three-dimensional plane groups G_{23} , and two-hundred thirty three-dimensional space groups G_{33} . The complete enumeration has been given in various sources E. Federov (1885), A. Schonflies, and W. Barlow in the 1890s.

3.3.5. *Lattices*

The set of the symmetry subgroups of a particular symmetry group can be further sorted by a relation that orders all the subgroups in the set. If this relation can be established for all pairs of elements in the set then this relation is called ‘total or strict order’ and the set is called ‘chain’. For instance, the relation “less than or equal to” ($=$) is a total order on integers, that is, for any two integers, one of them is less than or equal to the other. If this relation is defined for some, but not necessarily all, pairs of items, then the order is called ‘partial order’ and the set is called a ‘partial ordered set’ or ‘poset’. For instance, the sets $\{x, y\}$ and $\{x, y, w\}$ are subsets of $\{x, y, z, w\}$, but neither is a subset of the other. In other words, the relation “subset” is a partial order on sets. Formally, both total order and partial order are relations that are reflexive, transitive and antisymmetric: Reflexive is a binary relation R for which aRa for all a . Transitive is that binary relation R for which aRb and bRc implies aRc . Antisymmetric is a binary relation R for which aRb and bRa implies $a = b$.

One of the most useful features of ordered sets is that, in the finite case, they can be drawn. Relationships between subsets of a set will be pictured in two ways (Dean 1970): a) ‘Venn Diagrams’ where each set is pictured as a subset of the plane and the subsets of interest are shaded; and b) ‘Hasse Diagrams’ where each set is designated by a dot on the plane and the R are designated with lines.

Typically graphs or Hasse diagrams are used to represent such order and show the nested relations of the subgroups diagrammatically in lattice diagrams. In graph representation, an empty relation between elements is represented by a graph with vertices and no lines connecting them and a complete relation between elements is represented by a graph with vertices that are all connected one to another. Different types of graphs represent types of hierarchies such as strict order, hierarchical order or semi-partial order. The graph of the symmetry group of the square is given in Figure 3-5. The diagram consists of four levels which correspond to the four possible orders of symmetry subgroups that are divisors of the maximum order of eight of the symmetry group of the square. The top level depicts the complete symmetry group of the square, the second and third levels depict the subgroups of orders four and two respectively, and the last level depicts just the subgroup that contains one element, the identity. All ten subgroups are positioned within the lattice.

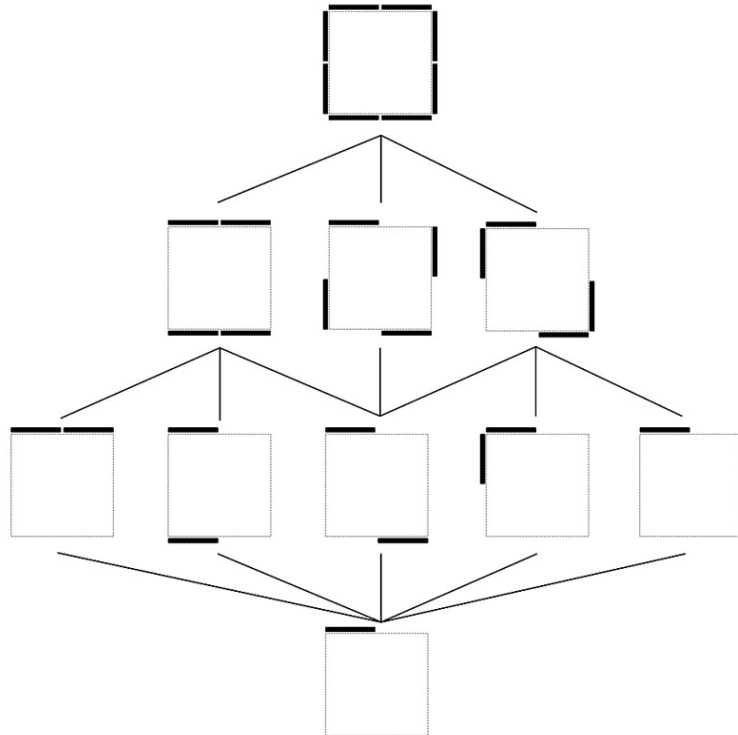


Figure 3-5: Order of sub-symmetries of the ten subgroups of the symmetry group of the square

3.3.6. Conjugacy

The structure of the symmetry can be further worked out by sorting out the symmetry elements and the symmetry groups with other types of relations. A significant relation is one that partitions the sets of symmetry elements of a group into equivalent classes of isometries that are characterized by the same type, i.e., they impose the same type of transformation or rearrangement within a spatial structure. This relation is called ‘conjugacy’ relation and it is an equivalence relation between elements. For example, in the case of the square, a reflection through the horizontal mid-edges is similar to a reflection through the vertical mid-edge but both are different from the reflections that pass through the diagonals of the square. The choice of the spatial system within which the conjugacy relations are defined affects the ways that the designer looks at a design. Different conjugacy relations result in different decompositions of designs and different orderings of the resulting subgroups (March, 1996a, 1996b, 1996c).

Formally, given elements x, y , of a group G , x is conjugate to y if $g^{-1}xg = y$ for some $g \in G$. The equivalence classes are called ‘conjugacy classes’ and the elements within the same class must have the same order. The conjugacy class of an element x in G is found by calculating $g^{-1}xg$ for every $g \in G$. Similarly the conjugacy class of a power of x , say x^m , is found by calculating $g^{-1}x^mg$ for every $g \in G$. By algebra, once some of the conjugacy classes have been found, the other are easily calculated. Figure 3-6 shows three conjugacy classes for the symmetry subgroups of the symmetry group of the square. The graph in Figure 3-6a is the complete enumeration of all possible symmetry subgroups; the graph in Figure 3-6b considers reflections through edges and vertices as distinct and the graph in Figure 3-6c takes all reflections as typologically similar.

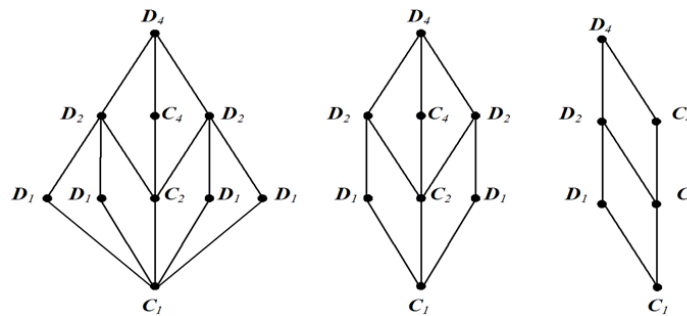


Figure 3-6: Partial order of the conjugacy classes of symmetry group of the square
a) 10 subgroups; b) 8 subgroups; c) 6 subgroups

3.3.7. Isomorphism

The structure of the group has been so far investigated through the products of the elements that comprise the groups and the relations order the resulting groups and subgroups. A very different direction of analysis is through the comparative analysis of diverse groups. One of the most interesting aspects of this new focus is the possibility of simplifying and the ability to recognize apparently different problems as basically the same. The relationship that allows it is called ‘isomorphism’. Formally, given two groups G and H , the groups are isomorphic if there is a bijection $f: G \rightarrow H$ such that for all $a, b \in G$ there is $f(a*b) = f(a)*f(b)$.

A specific class of bijections is quite interesting; the bijections between a set and itself are known as permutations of the set. A permutation is a rearrangement of objects. The collection of all permutations of K constructs a group S_k under composition of functions. If K consists of the first n positive integers, then S_k is written S_n and called the symmetric group of order n . The degree of S_n , that is the number of objects involved, is n . The order of S_n is $n!$ and the elements of the permutation group are the $n!$ rearrangements of the set. Permutations are nicely represented in the so-called ‘cycle notation’, whereas only the objects that are moved are written within a pair of brackets; in this notation a permutation $(x^1 x^2 \dots x^k)$ is called a cyclic permutation. The number k is its length and a cyclic permutation of length k is called a ‘ k -cycle’; a 2-cycle is usually referred as a ‘transposition’. In section 3.3.1, Table 1 shows all the multiplications of functions for the 24 bijections that form a group.

Permutation groups or substitution groups are of particular interest because they provide concrete representations or realizations for all finite groups. Furthermore, permutations are important in the study of symmetry itself since any symmetry operation is a permutation of a set. A symmetry operation is not an arbitrary permutation but rather one that leaves the structure under consideration invariant; for example the symmetries of the square may be viewed as well as permutations of the four vertices that keep the structure of the square invariant. Similarly the symmetries of the Euclidean space are those permutations of the points in space that preserve distance.

Cayley’s Theorem (Walker 1987) asserts that every group G is isomorphic to a subgroup of S_G , the group of all permutations of the set G . If G is finite and $|G| = n$, then S_G is isomorphic to S_n , the symmetric group of degree n . Any symmetry transformation can be considered as a specific permutation of a set that leaves the structure under consideration invariant. The decomposition of

a two-dimensional shape in its basic spatial elements consisting of points and lines provides the best members for this set. For example, in the case of the square, the elements of the set that may be permuted by the symmetry transformations can be its four vertices or its four edges. Vertices and edges are not the only candidates for this set; in fact a description of a two-dimensional geometrical figure in terms of its internal diagonals is considered by mathematicians as the most elegant and parsimonious one because it has the simplest structure; it involves a minimum set of elements which completely specify the properties of the structure under consideration. The three descriptions of the structure of the square are given in Figure 3-7.



Figure 3-7: Description of the square in terms of its vertices, edges, and internal diagonals

The elements of the symmetry groups of the square can be written down in the form of cycles of permutations of a set consisting of the vertices, edges or diagonals of the square. The sum of all combinations or products of cycles of permutations under the elements of the symmetry group divided by the total number of the elements in the group, is the ‘cycle index’ of the corresponding permutation group.

The complete computation of the cycle index of the permutation group of the vertices is given here for the square and is illustrated in Figure 3-8. There are eight distinct transformations that bring the square back to its original position and they are all isomorphic to eight permutations of vertices that leave the structure of the square invariant. The first transformation is the permutation corresponding to the rotation consisting of no motion at all. This permutation has four cycles of order one and is represented as f_1^4 . The next permutation corresponds to a clockwise rotation of 90° around the center of the square and is comprised of one cycle of order four represented as f_4 . The next permutation corresponds to a rotation of 180° around the center of the square and comprises the product of two cycles of order two, it is represented as f_2^2 . The next permutation corresponds to a clockwise rotation of 270° around the center of the square and comprises one cycle of order four represented as f_4 . The next two permutations correspond to the two reflections about the axes passing through the mid-edges of the square and each consists of a product of two cycles of order two represented as f_2^2 . Finally there are two permutations that correspond to the

two reflections about the vertices of the square and each is consists of a product of two cycles of order one and one cycle of order two represented as $f_1^2 f_2$. The complete computation of the eight products of cycles of permutations is given in Figure 3-8.

	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	→	$(1)(2)(3)(4)$	→	f_1^4
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$	→	$(1\ 2\ 3\ 4)$	→	f_4
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	→	$(1\ 3)(2\ 4)$	→	f_2^2
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$	→	$(1\ 4\ 3\ 2)$	→	f_4
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$	→	$(1\ 2)(3\ 4)$	→	f_2^2
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$	→	$(1\ 4)(2\ 3)$	→	f_2^2
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$	→	$(1)(3)(2\ 4)$	→	$f_1^2 f_2$
	→	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$	→	$(1\ 3)(2)(4)$	→	$f_1^2 f_2$

Figure 3-8: Complete visual computation of the cycle of permutation of vertices of the square

The sum of all compositions or products of the cycles of permutations under the elements of a symmetry group divided to the total number of the elements in the group is the cycle index of the corresponding permutation group of the vertices of the square. The cycle index C of the vertices of the square induced solely by rotations is given by the symbolic expression (1) and the complete index C of the vertices of the square is given by the symbolic expression (2).

$$C = f_1^4 + 2f_4 + f_2^2 \quad (1)$$

$$\underline{C} = f_1^4 + 2f_4 + 3f_2^2 + 2f_1^2f_2 \quad (2)$$

The relevance of the cycle index in general problems of combinatorics was first presented by *Polya* in his theorem of ‘counting non-equivalent configurations’ with respect to a given permutation group (Polya and Tarjan 1983). March (2002) has emphasized the value of this theorem in a design context and in contemporary architectural research and several examples for the complete enumeration of non-equivalent configurations in cellular automata and sound structures have been given by Economou (1998); (1999).

3.3.8. *Counting non-equivalent configurations*

The relevance of the cycle index in general problems in combinatorics was first presented by *Polya* in his theorem of ‘counting non-equivalent configurations’ with respect to a given permutation group (Polya and Tarjan 1983). Polya used to say ‘The cycle index knows many things’ (Polya et al, 1983, 67) and indeed this the case: Say that one wishes to know in how many ways one can assign three colors or features on the vertices of a square, provided that one uses one color per vertex, and one also wants to count as distinct the ‘enantiomorphs’; if one takes the cycle index of the permutation group of the faces of the oblong induced by rotations, and a figure inventory $x+y+z$ for three colors x , y , and z , and substitute the figure inventory into the cycle index by replacing $f_k = x^k + y^k + z^k$, then, by expanding the cycle index in powers of x , y , and z , the resulting coefficient of, say, $x^r y^s z^t$ is the number of distinct ways one can paint r faces with a color x , s faces with a color y and t faces with a color z .

The appropriate method for this inquiry has been given by Polya in his theory on counting non-equivalent configurations with respect to a given permutation group (Polya et al, 1983). Essentially, Polya's theory of counting specifies the numbers of different ways one can assign k qualities to n vertices of an n -cornered figure without considering any two arrangements as different if they can be transformed one to another by a symmetry operation. In this case, the

application of Polya's theorem specifies the different number of ways that $k = 1, 2, \dots$ colors can be assigned in all n similar parts of the blocks. Note however that Polya's formalism provides the answer even if $k > n$; that means that in principle it is possible to know the different number of ways one can paint, say, the four vertices of a square, provided that one uses one color per face, with any number of colors. The computation here limits the number of colors to two in order to present the computation in its simplest format. The figure inventory will be $x+y$. If one substitutes the figure inventory into the two expressions (1) and (2) by replacing

$$f_k = x^k + y^{k^2} \quad (3)$$

The corresponding cycle indices of the rotational and the complete permutation groups of the vertices of the square are expanded in the expressions (4) and (5).

$$C_v = ((x + y)^4 + 2(x^4 + y^4) + (x^2 + y^2)^2) / 4 \quad (4)$$

$$\underline{C}_v = ((x + y)^4 + 2(x^4 + y^4) + 3(x^2 + y^2)^2 + 2(x + y)^2(x^2 + y^2)) / 8 \quad (5)$$

The expansion and computation of both symbolic sentences in (4) and (5) can be done with the binomial theorem given in (6).

$$(x+y)^n = \sum_{r+s=n} \frac{n!}{r!s!} x^r y^s \quad (6)$$

The details of the computation are not given here but are left to the interested reader. The result of the computation is given in (7).

$$\underline{C}_v = (8x^4 + 8x^3y + 16x^2y^2 + 8xy^3 + 8y^4) / 8 = x^4 + x^3y + 2x^2y^2 + xy^3 + y^4 \quad (7)$$

The coefficients of (7) give the numbers of non-equivalent configurations one can get using the structure of the square in a basic format of 2×2 . The equation is symmetric with a vertical axis which means that results are the same for, say, the configuration of four white squares (x^4) and

the four black quadrants (y^4). The computation also states that there should be two non-equivalent ways of arranging two white and two black quadrants (x^2y^2) upon the structure of the square. All non-equivalent configurations are given in Figure 3-9.

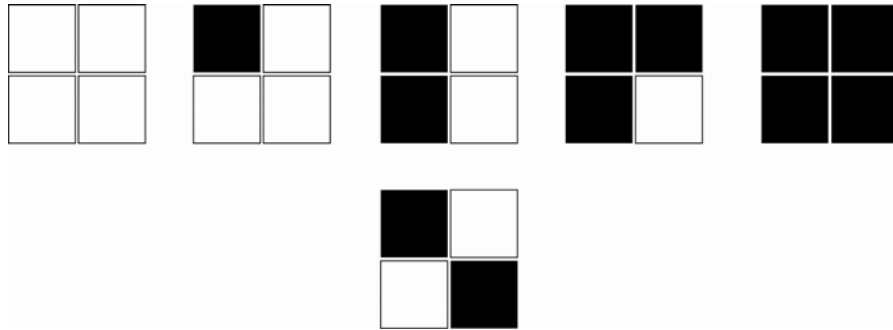


Figure 3-9: Non-equivalent configurations based on the symmetries of the square

3.4. Tracing histories

‘Do not be misled by the appearances. Things which look different may have the same meaning’. Al-Fullani (1732)

The history of the development of the fundamental mathematical concepts of group theory is an integral component of the development of mathematics in late eighteenth and nineteenth century. The origins of the concept can be traced all the way back in the third millennium BCE to the symbolic systems found by the Sumerians and their studies in simple arithmetic (Neugebauer 1957). The addition table was the first act of abstraction that changed the meaning of addition. What the addition table does is assigning a definite number called a *sum* to every ordered pair of numbers. It would take several centuries of constant development and redevelopment of mathematical thought in various domains number theory and algebraic structures to eventually see addition as one of the earliest and most profound acts of mapping. A mapping is a function that assigns to each ordered pair of objects in a set another object from that set and the particular function encountered in the addition table of the Sumerians is called a binary operation. Similarly, Sumerians separated the operation of multiplication from its original meaning of finding the cardinal number of a rectangular array of objects by mapping ordered pairs of natural numbers into the systems of natural numbers. Two more gifts received from the Sumerians are the place

value system, that is, the transposable meaning of a digit depending on its position in the written numeral, and the identity, what everyone calls nowadays the zero element in addition and the unity element in multiplication (Neugebauer 1957). It would not be farfetched to fathom that Sumerians could represent numbers as points on a line, and therefore present an early species of the analytic geometry. However, the word number continued to refer to counting or measuring because of the isomorphism between cardinal and natural numbers. The true divorce will be marked later by the creation of number systems without numbers with the theory of sets to access a theoretical space. The task of the Cartesian thought was the shifting of this old paradigm towards analytical geometry. Out of the symbolic thought, it became clear that all of the knowledge of space and spatial relations could be translated into a new language of number system without numbers. Through these processes of translation and transformation, the true logical character of mathematical thought could be conceived in modern times.

Clearly, this history of the origins and development of group theory can never be claimed to be comprised of a unique trajectory; what really emerges through a sum of several trajectories is that each illuminates some aspects of the theory. Particularly interesting among them are diverse strands emerging in various parts of the non-western world, and occasionally peripheral in the compilation of this history. Some of these stands can be accounted as predecessors of group theory in non-western world and include cases such as: a) the numerals from the Sahara civilization transmitted to Europe through Spain in the tenth century CE (Smith and Ginsburg 1937); b) the oldest piece of chessboard found in Mohenjo-Daro, capital city of the Indus civilization (Canby 1961); c) the rational approximation of the diagonal of the square $\sqrt{2}$ in the eighteenth century BCE (Neugebauer 1957); d) the arithmetic formula of the truncated pyramid given in the Rhind papyrus (Banchoff 1990); e) the binomial expansion, known as the Pascal triangle shown in The writings by al-Maghribi in the twelfth century CE (Ifrah 1985); f) the geometric resolution of cubic and quadratic equations by means of intersections of circles, parabolas and hyperbolas mentioned by Omar Khayyam in his Algebra book, in the twelfth century CE (Sesiano 2000); g) tic-tac-toe arithmetical games practiced in Monomotapa, Africa by the seventh century CE (Zaslavsky 1973); h) geometric algorithms of the kind of Euler's Konisberg bridge manifested in the mukanda initiation rites in the Central African kingdoms since the fourteenth century CE (Gerdes 1999); i) magic squares practiced in the Islamic world since the tenth century CE – see 'Harmonious Dispositions of the Numbers' in al-Antaki's Book III, algorithmically defined by Al-Fullani al-Kishnawi al-Sudan, a native of Nigeria in the eighth century CE; and many others too.

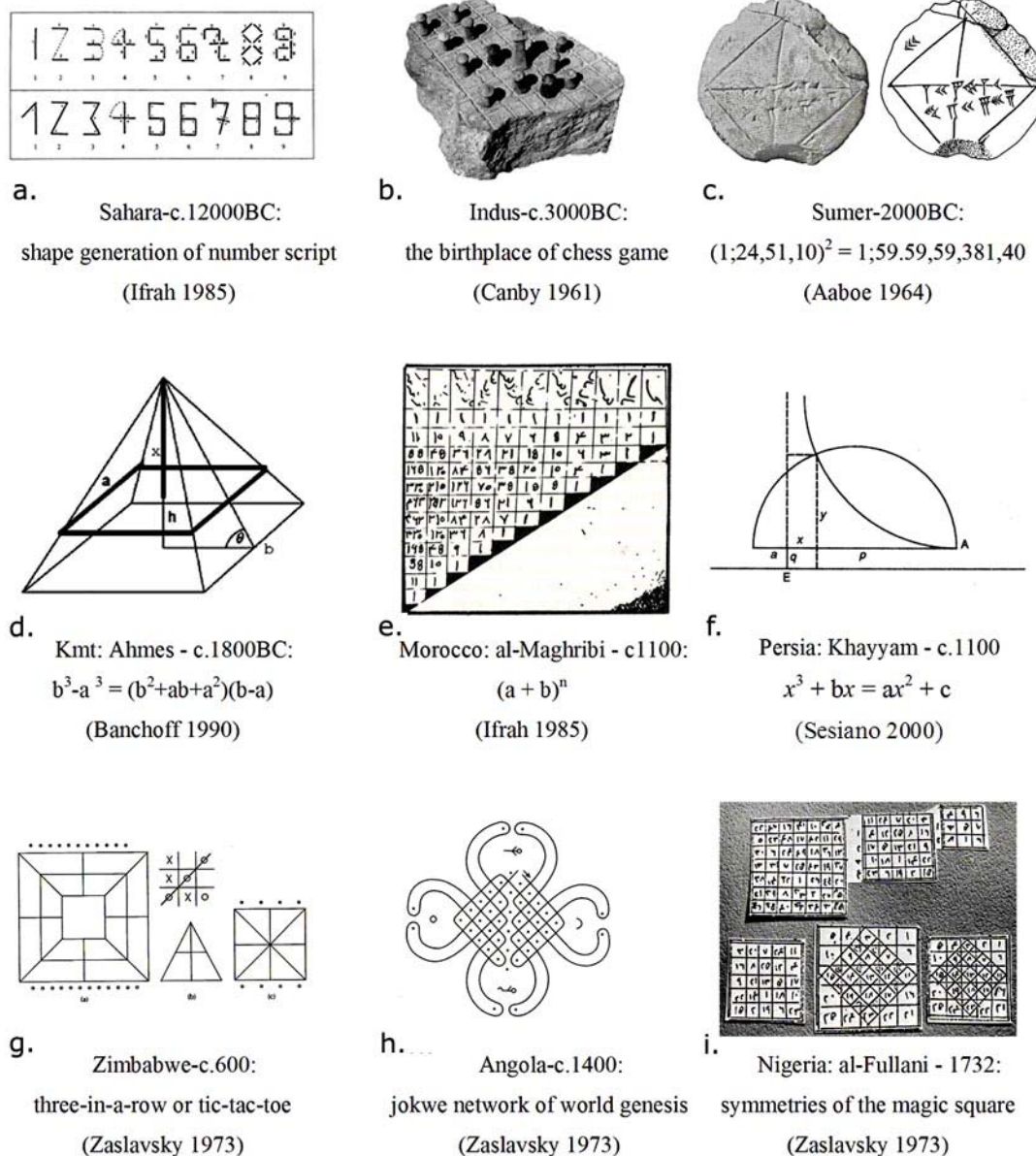


Figure 3-10: Patterns exemplifying group theory applications in the non-western world

The core of group theory as it is understood nowadays was developed primarily during the eighteenth and nineteenth centuries out of the confluence of several and diverse investigations in fields of algebraic equations, permutations, number theory and others. Similarly the contemplation of the applicability of the emergent group theory to describe mathematical and physical aspects of space was also the product of these times. The beginning of the modern histories can track back to Monge's 'Descriptive geometry' (1794), the language of the engineer, an entirely new language that had as a task to shape objects by an exact measurement of their

geometric properties. The work of Poncelet (1812-14), in his development of projective geometry, clearly built upon the work by Monge and constituted the foundations of the group theoretic approach in geometry, along with the descriptive geometry. This approach is particularly built around two issues: a) the elimination of the metric from geometry and b) the extension of the coordinate concept. The elimination of the metric information from geometry was achieved primarily by the dissociation of metric properties from the incidence properties. Poncelet, in his work on projective geometry, anticipated the analytical treatment of geometric figures, that is, the shift from synthetic projection to the analytical study of coordinate transformations in search of invariants and made it possible to apply invariant theory, rooted in number theory, to classify geometric objects. The extension of the coordinate concept was achieved by the shift of the meaning of coordinates from intervals to numbers. A coordinate system of a geometric manifold consists of independent parameters. Therefore, a space becomes a number manifold and this view of space separates the study of objective physical space from the study of mathematical spaces, and of physics from geometry.

The development of non-Euclidean geometries articulated even better this new vision of abstract, transformational geometries. The development of hyperbolic geometry independent of the parallel postulate of the Euclidean geometry ran into the epistemological problem of space. In order to separate geometry from physics, Riemann (1854) used the term ‘space’ to denote objective physical space and ‘manifold’ to denote mathematical space. The turn toward abstraction was completed with the introduction of n-dimensions. Gauss promoted the theory of algebraic equations and Lagrange in 1770 tried to determine why the solutions of cubic and quadratic equations work. Developing an approach of the combinatorial calculus type, he anticipated the subsequent permutation-based theory of solvability of algebraic equations. Cauchy by 1815 played a central role in shaping permutation theory. He elaborated the terminology for the concepts which one now calls group, order of a group, index of a group, and subgroup. By such an arrangement, Cauchy meant an ordered string of quantities. A permutation or substitution denotes a transition from one arrangement to another (Wussing 1984). Galois in 1831 established that the algebraic equation $f(x) = 0$ of degree p is related to the structure of a group and that there is a connection between the solvability conditions of algebraic equations and permutation theory. By 1832 Galois reached the fundamental concept of the normality of certain subgroups but his work remained unknown until published fifteen years later by Liouville (1846). The Galois theory is regarded now as a ‘show piece of mathematical unification, bringing together several different branches of the subject and creating a powerful machine for the study of

problems of historical and mathematical importance' (Stewart 1992). The fundamental theorem of Galois theory establishes the correspondence between groups and fields.

The development of the concept of a permutation group marked the first stage in the evolution of the abstract group concept. Jordan's (1870) treatise must be regarded as crucial with its attempts to synthesize arithmetic and geometry by means of the permutation theoretical concept of a group. Closure under multiplication is declared then as the sole property required of a group (Wussing 1984). That is the case both in the definition of a group and in the presentation of Galois Theory. Cayley (1845) published his ideas on permutations and provided remarkable insight on the abstract conception of a group as a system of defining relations. The theory of invariants yielded the long-sought tool with which to bring to light connections between metric and projective geometry. Cayley (1859) used what is now known as the Cayley metric to embed Euclidean metric geometry in the general scheme of projective geometry, then came up with the concept of 'distance', defined as every relation that satisfies the condition Because of his involvement with the determination of systems of invariants, Cayley did fail to discover the connection between the metrics and non-Euclidean geometries. At the time after Jordan's treatise was published, geometry came to be "a new attraction to the theory of permutations" (Wussing 1984). The decisive moments for the post-1870 evolution of the abstract group concept are those when the permutation-theoretic group concept invades geometry, leaving permutation theory behind.

The major catalyst for the unification of the various studies in group theory and its applicability as a fundamental construct for this unification of many and diverse geometries is really the Erlangen Program of 1872 by Klein. Klein in 1870 embarked on metrics associated with all types of quadratic curves and quartic surfaces and came up with plane and solid hyperbolic and elliptic geometries. Klein in 1871 wrote: "I wish to construct plane and space representations of the three geometries (Euclidean, Hyperbolic, and Elliptic) that would afford a complete overview of their characteristic features" (Klein 1921). Klein contributed to the formulation of the concept of group of transformations by forcing the transition to the explicit thinking in terms of groups. Logically and historically, there is a distinction between the use of group theoretic reasoning in geometry and the use of motions or transformations as group elements. The use of the group concept, in the form of group of transformations, for the purpose of classification in geometry brought a modified notion of motion in geometric thinking. The relation between physical motion and coordinate transformation shifted in favor of the physical view. This train of thought led to the idea that mathematics associated with motion must be pursued as the study of groups of motions

and that the study of space could be facilitated by such as a framework of thought. Riemann and Helmholtz had attempted earlier to axiomatize geometry; if geometry is the structure of objective space, then it could be described in terms of the possible motions of physical bodies.

The linking of groups of geometric motions with their generators produced the advance that enabled Klein to apply the fundamental principles of permutation theory to geometry and to work out the concept of a discrete group of transformations. Furthermore the shaping of the abstract group concept by Cayley pointed towards an abstract view of groups, obviously influenced by the abstract position of Boole whose ‘An Investigation of the Laws of Thought’ was published in 1854. So, when Cayley came back to group theory in 1878 with his ‘Theory of Groups’ illustrated with the graphical representation of the groups, he stressed the role of generators, and received the long overdue recognition. The last pages of this initial theoretical grounding of group theory were written by Dyck (1882) and Burnside (1897). Dyck, one of Klein’s students, took advantage of the parallel development of mathematical logic and wrote his ‘Studies in Group Theory’ that completed the elaboration of the abstract group concept. His concept of a group, remarkable for its historical objectivity, fulfilled all the requirements demanded of a fully developed abstract approach. Burnside’s ‘Theory of Groups of Finite Order’ (1897) continued the study of group theory and prepared the ground for its applications in various fields and in particular in the study of symmetry.

3.5. Summary

The mathematical language of symmetry introduced in this chapter included all the fundamental concepts of group theory such as group axioms, elements, composition, associativity, identity, inverse, commutativity, structure, and order of a period of element. The analytical description of the structure of a group was given in terms of the concept and properties of a multiplication table and the constructive description of the structure of a group was given too in terms of the concepts of group generator, set of group generators, and sets of defining relations. The pictorial description of the structure of the group was presented including the concepts of the graph of group, Cayley diagram, directed network, correspondence of group element \leftrightarrow graph vertex, group generator \leftrightarrow graph directed weighted edge, group word \leftrightarrow graph path, group composition of elements \leftrightarrow graph succession of paths, and group identity word \leftrightarrow graph closed path. The

concept of subgroup was introduced including Lagrange's theorem and its relation to group generators and cosets. The concept of lattice representation of subgroups within groups was introduced including ordering relations such as strict order, hierarchic order, and partial order. The concept of conjugacy class was introduced along with equivalence class, conjugate elements, conjugate subgroups, and partial order of conjugacy classes. The idea of isomorphism was introduced with respect to the permutation groups the permutation groups and a basic account of Polya's theorem of counting non-equivalent configuration with respect to a given permutation group was given in the end.

The comprehensive review of group theory and symmetry presented above was structured around the square to provide a consistent set of parallel descriptions. It has helped to bring forward the logical framework upon which a significant body of research in architecture lies upon. Our path from the past to the present has opened up the issue of how a post-Cartesian shift on representation has dismantled old ways of see things. By developing a consistent treatment of the structure of the square through symbolic, theoretical and abstract representations, the groundwork for the theoretical model has been prepared for further development in this research.

CONTRIBUTION

Chapter 4 Abstraction, projection, weighting, layering

Chapter 4 provides the hypothesis and methodology of the dissertation. Currently all formal analyses using group theoretical tools focus on repetitive designs that show immediately their symmetrical structure. It is suggested here that highly complex designs can still be described and analyzed in a group theoretical manner. The key idea is that the complexity of these designs can be seen as an aggregation of spatial layers that can all be decomposed by subgroup relations found within the symmetry of the underlying configuration.

4.1. Introduction

‘In architectural projection space is nothing more than pictures of light... Plan, section, and elevation, considered independently, are almost prehistoric. They can exist, and even coexist, without invoking projection (or indeed light) at all. A plan need not be regarded as a picture; it can just as well be thought of as a set of geometric operations on a flat sheet ‘. Evans(1995)

A fascinating aspect of certain classes of architectural works is their ability to escape easy interpretations based upon existing formal tools. This is especially true for several architecture works of the modern movement that feature all kinds of asymmetrical arrangements. It is claimed here that a new look at existing formal tools can perform the task, if applied, though differently. The key idea here is that spatial representations of complex objects can be understood as layered compositions of simpler parts that can pictorially illustrate the symmetry structure of such a spatial configuration. It is argued that if a specific schema is chosen – say a square or a triangle, or else – then the maximum number of layers relevant to that configuration can be built upon the number and qualities of the symmetry subgroups of that spatial structure. The key idea suggested here is that the use of partial order lattice as a specific construct of formal analysis for the description of designs, along with finer distinctions of representation to account for patterns of ambiguity and emergence in the description of space, including weighting for depth and projection for transparency, can capture several properties of the description of designs that would otherwise be unnoticed. Even stronger, it is claimed here that the specific formal construct proposed here can address otherwise informal and intuitive properties of formal composition

typically subsumed under the headings of balance, rhythm and proportion in a very specific and formal way.

4.2. Representation

'Representation has to do with the way the objects in a computation are described... Representation is usually divided into the verbal kind and the visual kind. The verbal kind is logical and scientific. By contrast, visual representation is characterized by the lack of primitives and by a corresponding vagueness in presentation.' Knight and Stiny (2001)

The key ideas about the representational model are developed here. It is suggested that four aspects of representation are paramount for the formal analysis proposed here. These aspects are: a) abstraction; b) projection; c) weighting; and d) layering. Furthermore, it is suggested that these representations can be combined with one another in a group theoretical manner to accurately describe specific properties and characteristics of the formal structure of an architectural work.

4.2.1. Abstraction

'What in art is called 'abstract form' ... is actually concretized conception.' Babichev in Senkevitch (1983)

In architectural design, a line drawing provides an abstracted representation of an imaginary or a real architectural object. This visual representation does not depend solely on formal similarities. This representation is not a mapping of an object's complete form but a mapping of certain privileged or relevant aspects. Depending on the number of features being deleted from the original or alternatively being added or transformed in the mapping a level of abstraction is then arbitrarily defined. Still this is a contested territory. A typical mapping between an object and its representation is often understood as a distinction between concrete and abstract. Arnheim (1969) reworks these definitions regarding the dichotomy between concrete and abstract: 'The abstract objects of thought, such as numbers, law, or perfectly straight lines, are real parts of nature even though they exist not as particulars...'. In gestalt theory perception of shape starts in a grasping of

generic structural features (Arnheim 1969). Perceptual organization enlists invisible extensions as genuine parts of the visible. In this context human perception is envisioned as a unitary process which leads without break from the elementary acquisition of sensory information to the most generic theoretical ideas. The essential trait of this unitary cognitive process is that at every level, it involves abstraction.

Here abstraction is used entirely in terms of essential and accidental properties (Descartes 1961). An abstracted version in any mapping is the one that keeps certain characteristics of the original object while dropping others. All such different levels of representation allow the analysis to describe shapes at different generative stages and establish links which are not immediately available to the viewer (Flemming 1990). Several types of abstraction representations are shown in Figure 4-1.

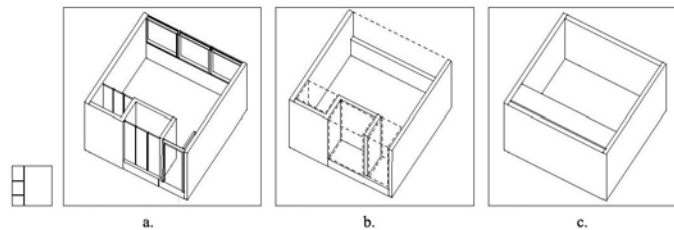


Figure 4-1: Different levels of abstraction

4.2.2. Projection

‘Projection is thinking of something as having properties it does not have, but that we can imagine without being conscious that this is what we are doing. It is thus a species of thought – thought about something’. Putnam (1987)

A fascinating aspect of architectural representation is its ability to describe architectural space through the deployment of depthless drawings. This paradigm shift occurred primarily in the first decades of the sixteenth century and it is exemplified by several cases where architectural projections including plans, sections, interior elevations, exterior elevations and so forth are all joining in their corresponding parts by parallel lines. These lines are ‘the agency through which the space outside the surface of the drawing is brought into it’ (Evans 1995). Thus architectural projection, becomes ‘nothing else other than the finest light’ (Panofsky 1968); the images drawn as if transmitted to a surface by light appear flattened into a comb of drafted lines. Since then, light has become the ultimate geometric instrument, and lines are identified to light paths (Evans

1995). The geometry of images propagated by light has since been developed as a postscript of that of land survey.

More importantly, the unintended consequence of all these linear connections of representations was to create a new kind of representation that grasped the imaginary space behind the original drawings and opened up to space. This showdown, obscured by an architectural space limited to the pictures that gave access to it, finally opened up to the parallel, orthographic projection of the architecture that brings space into pictures. Since then, a plan ceased to be regarded as a picture and came into being as a set of geometric transformations on a flat sheet.

Significantly, these initial computations with shapes were not about descriptions of forms of buildings as extant constructions of physical materials in physical space. Rather, they were descriptions of designs to become ‘constructions of imagination’ (Mitchell 1990). In this way, these initial computations opened up the way to think nowadays of the space populated by collections of graphic tokens such as points, lines, and polygons as two-dimensional and three-dimensional design worlds. And among all these worlds, the design world consisting of lines in the plane, is assuming an authority that maps directly from Alberti’s world directly to the world of CAD systems.

Still these lines – and all lines in any spatial system – come with a baggage; they stand for orthographic and perspective projections and the choice of the system is often ambiguous. Here the main concern deals with a specific kind of combined orthographic projections through descriptive geometry and in particular the specific composite drawing technique consisting of orthographic and affine transformations. This representation is quite unique in architecture and not in other engineering fields, for of its ability to represent abstract space both visually and metrically. This type of composite projection consisting of orthographic and affine transformations produces what is nowadays understood as ‘paraline drawings’ both axonometric and oblique (Uddin 1997).

Paraline drawings are capable of creating angular variety and having different emphasis, using the same plan and elevation-section. In an oblique projection, the surface parallel to the picture plane remains to its true size while the other two visible surfaces of the object become foreshortened. Paraline drawings offer a suitable alternative to perspective and orthographic projection because they combine plan, elevation and section in one three-dimensional drawing where lines are drawn to a scale. Axonometric projection refers to the image being foreshortened

as a result of inclined and rotated portion of the object with respect to the picture plane. In contrast, the desired appearance of an oblique projection depends on three factors: orientation of the primary orthographic projection, angle and direction of receding lines, and ratios of foreshortening for vertical planes. This type of parallel projection is efficient for complex architectural representation, due to its versatility, flexibility, and capability to show more information and support designing more than any other medium. Several types of axonometric and composite axonometric views are shown in Figure 4-2.

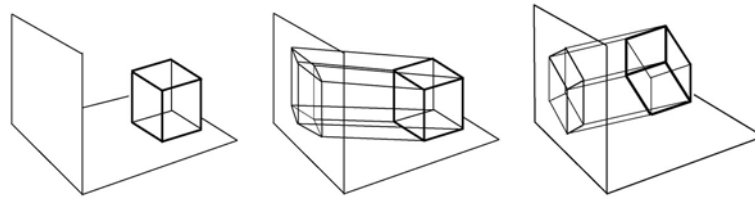


Figure 4-2: Types of axonometric projections – a) initial state; b) plan oblique; c) elevation oblique

The nature of the images generated by parallel drawings attracted the interest of twentieth century architects because seeing in oblique views is essential for depth perception. It is significant that the neo-plasticists and the suprematists manufactured the comeback of parallel projection into the central stage of modern architecture. It is also with no surprise that the revival of the 1920s in the 1960s relied upon architectural axonometric as the primary medium of representation and it was carried to new heights by the New York Five (Deamer 2001) and their interpretations of the organizations of classical vocabularies of architecture including aspects of rectangularity, planarity, axuality, symmetry, and frontality (Evans 1995). It is significant as well that these latter appropriations of composite axonometric projections suggested two functions initially thought as one: first, the axonometric as a presentational device, and second as a conceptual device. The former is progressively enriched by the methods of multi-media presentation, the latter is able to support the communication of spatial qualities of an architectural object during the process of conceptualization. In all cases the exploitation of the fundamental ambiguity of the axonometric projection has been the discourse of much architecture in the twentieth century and it is suggested here that this should be the primary mode of representation to adequately describe and analyze the architecture works of these corresponding periods.

4.2.3. Weighting

‘The physical quality involves the visual effect of an appearance of weight and mass, or the force of gravity, acting on a form.’ Ladovsky (1926)

Weights are familiar in architecture. A pen stroke has a width. Lines are of different thickness. Typically these weights refer to what is represented and work upon a set of conventions widely shared and understood. Different types of conventions and assumptions often give the clues about what is represented. For example, Dokuchaev suggested that three types of formal manipulation can provide perceptual clues about the apparent mass and weight of a form: surface treatment, surface details, and formal allusion (Senkevitch 1983). In the first type, the surface treatment consists of the use of smooth surfaces to convey a feeling of lightness and roughly textured surfaces to make a form appear more massive. The second type involves the use of surface details, ranging from the fluting on a column to the joints in a wall and layering and interpenetration of planes in the surface of an abstract form. The third type of manipulation, formal allusion, is directly toward affecting the appearance of the form as a whole. Such manipulation ranges from the use of a single form, such as an inverted cone, to suggest the movement downward of a single force, to the conscious deformation of a simple or compound form.

Klee (Spiller 1961) and Stiny (1992), albeit in different ways, set the emphasis in the representation itself rather than what is represented. In this discourse, weights distinguish formal attributes from physical properties. Shapes made up of basic elements and weights answer to the Vitruvian categories: physical properties being included in firmness, functional properties in commodity, and spatial properties in delight. Several types of weighted representations are shown below in Figure 4-3a.

The lines, as apparatus, enable other conceptual functions. In architecture, the line is the means by which architecture displays its conceptual accretions. The intersection of the linear economy with a built form is Alberti’s remarks on the architectural wall (Alberti 1955). The wall is represented in the architectural section as a double line, as a path. Note that the edge is simply represented by a single line. Sectioned solid walls are mapped to solid | thick lines; in view, standard | thin lines outline contours; projected, contours are represented by single line, while close edges (beam) are represented by double dotted lines. Beneath this presentational level, more investigation is needed in the conceptual level. Meier’s drawings – plans, sketches, and working

drawings – are scalar analogs that serve to visualize or predict the experience of architecture, and are highly abstract notational scores that condense amounts of information – both visual and non-visual – into a codified language of symbols. A line on a plan may mark the separation of inside and outside, but it can also signify the edge of a volume, a change in material or level, or the presence of something above or beyond. Or it may indicate something not physical at all: axis of view, an alignment in space, or trajectory of movement. Solid, void, or glazing can all be noted with a limited catalog of linear marks. In part the clues to read these marks are contextual, fixed by the local syntax, and in part they are conventional, based on architecture’s accumulated memory.

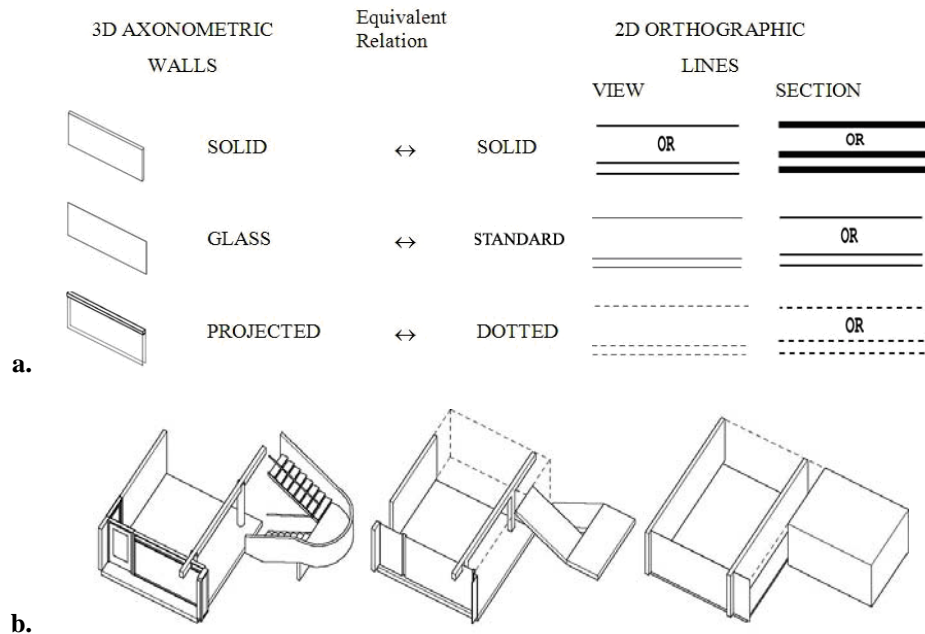


Figure 4-3: Types of weighted projections – a) Line weight notation; b) Axonometric line weight

This particular understanding of weights as a characteristic of representation itself is taken here as a powerful construct for formal analysis. In this latter view and according to Klee, weights suggest a different kind of projection, whereas ‘the irregular projection consists in the accentuation of parts or the omissions of certain parts’ (Spiller 1961). This type of representation directly alludes to transparency, a key feature of the twentieth century architecture, whereas the three-dimensional qualities of the design are illustrated through the weighting of lines. Varying degrees of transparency and opacity can be strategically deployed to hierarchically differentiate

preselected conditions important to the development of the work. The axonometric projection provides an opening in the hands of say, Lissitsky (1968) in his Prouns or Van Doesburg in his Counter-Compositions (Padovan 2002), that enables an access to a new kind of space proper to the new era by exploiting the ambiguity of spatial registration. The representation suggested here uses line weights as an integral part of the underlying mechanism of representation to adequately describe and foreground transparency in architectural works.

4.2.4. Layering

'Space, not stone, is the material of architecture. It is in space that the soaring wonders of modernity will be built by art plus the intellect.' Ladovsky (1920)

Spatial form is often conceived as a composite of layered planes making up its envelope and giving form to characteristics. The envelope of an architectural form possesses actual depth, like the wall of a building, and bears a certain correspondence with the idea of building up form in parallel vertical layers and to the dynamic planar constructions. Several examples would do but among the different architectural discourses the constructivism in particular abounds with exemplary case studies - see for example, Tatlin's counter-reliefs (Senkevitch 1983). This aggregation entails building up the envelope out of a network of layered planes, overlapping and intersecting at various coordinated angles. The envelope would thus acquire not only an essential three-dimensional aspect, but also a plastic quality, obtained from the interaction of receding and advancing planes and the rhythmic flow of their lines. All planes derive their ultimate impact and significance from being combined and arranged in equilibrium to create an integral architectural whole. The articulation of surface and form as envelope reveals how crucial the geometrical articulation of form was to the attainment of a lucid equilibrium: the shapes, sizes, and proportions of the constituent planes and intervals all had to be balanced and correlated with absolute precision.

This interpretation of a design as a system of boundary elements leads to a nested hierarchy of representations that start from definitions of bounded envelopes, continue to permeable envelopes and end to architectonic envelopes. The wall in all its instantiations is a prime determinant of spatial form and provides a major key of the solution to comprehend complexity. Fundamental problems here are found in both passages from one level of representation to the next. For example, the articulation of the permeable envelope as a prime determinant of spatial form is

inextricably linked to the prior definition of the boundary elements. The concept of permeability or alternatively the degree of a form's openness to circulation plays a major role in composition. The rationalists saw a dynamic level of permeability and the interpenetration of space that it suggests as a vital means for emphasizing the esthetic and perceptual power of space. Appropriately, the articulation of an architectonics envelope as an equally significant prime determinant of spatial form is inextricably linked to the prior definition of the permeable elements.

Here two interpretative ways of the wall function as a distributor of the forms of architecture are briefly discussed: the 'articulation in relief' (Ladosky 1926) and the 'organizational scaffolding' (Deamer 2001). The articulation in relief is the constructivist device to enclose the volume of the wall as a composite of layered planes. This technique has been proposed for constructing the envelope as an aggregate of planes, and is what Ladovsky calls articulation in relief. That kind of envelope possesses actual depth and bears Hildebrand's (1893) idea of building form in parallel vertical layers. This idea stresses the aesthetic of perception where the essence of architectural solution leads to 'the controlled modulation of spatial magnitudes' through means of expression such as mass, weight, color, proportion, movement, and rhythm. Another idea coming from Tatlin's (1915) counter-reliefs seeks to integrate into a spatial whole space within the construction and space beyond. Krinsky's (1921) Tribune Project materializes that approach.

On the other hand, the organizational scaffolding can be seen as the primary conceptual device of late-modern architects (Deamer 2001). It emphasizes the surface as an updated version of Gestalt theory articulated by Slutzky's painterly two-dimensional surface and in particular his work on phenomenal transparency depending on two-dimensional surface (Deamer 2001). Equally important is the work of their successors: it is the surface Libeskind "writes" on, it is the surface the computer turns into folds, it is the surface that advanced digital techniques turn to digital landscapes and topographies (Deamer 2001). Analogously, for the New York Five architects, this scaffolding to distribute the forms of architecture is the implied grid that locates the datum, the regulating lines, the location of frontal or rotated plane or surface, and the layering (Deamer 2001). For the current computer-based architecture, the scaffolding is still an a priori plane or surface that pre-determines the distribution of forms, thereby sparing the architect the need to make any arbitrary move. An example of layering mapping is shown in Figure 4-4. In this example, a decomposition of a design in terms of some parts in an initial representation parses the design in constituent parts that are all keeping their independence in any other representation.

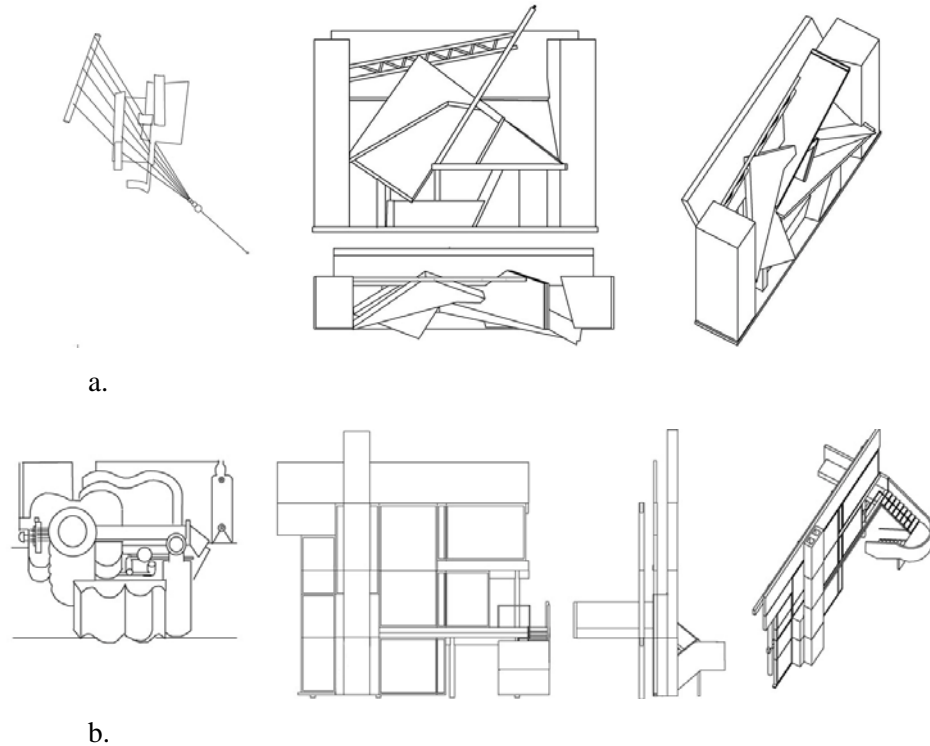


Figure 4-4: Two types of schematic drawings showing layered mappings. a) Tatlin and Krinsky's counter-reliefs: sculpture and facade; b) Jeanneret and Meier's scaffoldings: painting and façade

4.3. Partial order

'The semi-lattice is the structure of the complex fabric; it is the structure of living things – of great paintings and symphonies.' Alexander (1965)

All interpretations of designs suggest a decomposition of perceivable units and fundamental elements used or observed in their construction. Often these elements and units are identified with boundary elements and their arrangements, in particular walls and their assemblies. Here the key operator that parses the representations discussed above is the partial order relation defined by the symmetry group that describes the maximum symmetry of the configuration. The fundamental significance of symmetry arises here from its capacity to reveal two opposing aspects of form: transformation (change) and conservation (invariance). That which is conserved during a change is an invariant; the set of transformations which keeps something invariant is its symmetry group. The set of elements and their structural relationships forming the complete system are conserved

as a single whole and this order identifies all the nested parts in any configuration that have a group theoretical relationship to the overall group of the configuration.

The key idea is that spatial representations of complex objects can be understood as layered compositions of simpler parts and these parts can all be related through symmetry values from group theory. These values can be structured as a partial order lattice that pictorially presents the symmetry structure of any spatial configuration; the number and qualities of the symmetry subgroups found in any given configuration provide the maximum number of layers that can be found in a spatial configuration; for example, in any spatial arrangement that is based on the structure of the square the maximum number of layers and spatial constructs that can be build upon those is ten because this is the number of symmetry subgroups of the square. Figure 4-5 shows the partial order of the square.

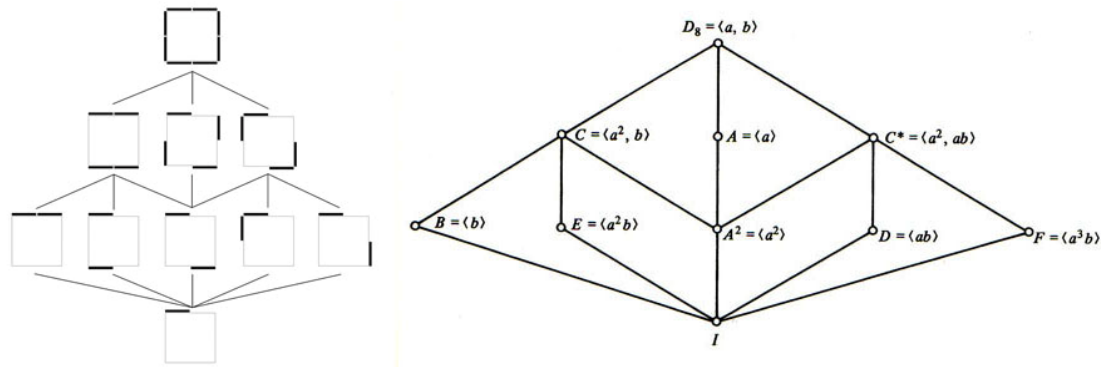


Figure 4-5: A partial order lattice of the square: a) set notation; b) discursive notation

4.4. Model

'Architects do not make buildings; they make drawings for buildings.' Evans (1995)

The key hypothesis is that spatial representations of complex objects can be understood as layered compositions of simpler parts and these parts can all be related through symmetry values from group theory. These values can be structured as a partial order lattice that pictorially presents the symmetry structure of any spatial configuration; the number and qualities of the symmetry subgroups found in any given configuration provide the maximum number of layers that can be found in a spatial configuration; for example, in any spatial arrangement that is based on the structure of the square the maximum number of layers and spatial constructs that can be build

upon those is eight because this is the number of symmetry subgroups of the square. Still, the symmetry subgroups can only provide the logical framework to compute an architectural composition; what is critical is the representation of the designs that are going to be analyzed within this framework. The model suggests four parts: a) a stop mode (\square); b) a rewind mode (\ll); c) a play mode (\gg); and d) a forward mode (\gg) and all four are discussed below.

4.4.1. Stop mode

The stop mode freezes the design in one representation that provides the blueprint for the formal analysis. This is often a contested ground as design is rightly considered as an element that emerges among many different kinds of descriptions and there is no ultimate description of a design (Stiny 1992). Here this initial design is considered to be the mapping of a collection of descriptions in a three-dimensional geometrical model that complies with the given drawings, in the fullest possible degree. The rendering of the model is then taken in a transparent axonometric view as a medium of architectural projection.

4.4.2. Rewind mode

The rewind mode entails an arbitrary number of successively abstracted models based on the initial three dimensional model of the stop mode. In principle there can be an infinite number of abstracted models that are successively removed from the original one. In theory there is a need for only one more additional model to show the transfer of the parsing of the one model to the other. Still here, two additional levels of notational languages are suggested as a minimum for formal analysis to comprise along with the initial one a set of three distinct notational languages. The first level of notation or the stop mode, is the architectonic level, or the level that approximates in some way the original notation used for the language of the actual design model into its closest geometric representation. The notation privileges functional elements such as walls, slabs, columns, and beams, walled furniture, handrails, and openings of various kinds such as windows, doors, stairwells, chimneys, and so forth. All elements are weighted and usually represented as architectural conventions indicate. A weighted model contains the whole set of architectural features, i.e. a collection of information with values made explicit by the designer. Wall tectonics, physical properties, spatial functions and all kind of features to be expected in design models. This level concretizes the geometric features into their material expression

(Alberti 1955). The units of the surface structure belong to a repertoire that constitutes the visual vocabulary of the designer.

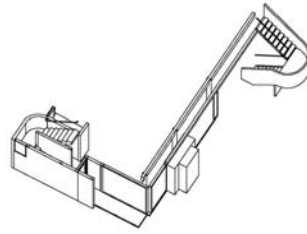


Figure 4-6: Example of architectonic notation

The second level of notation, the spatial level, privileges space divisions and corresponding openings in these boundaries, and discards all other information. This level essentially picks up planes that function as walls and slabs and so on, the connections between them and their interface with context for ventilation, light and so forth. At this level, a spatial model that emerges is a spatial decomposition of the building with geometric shapes that bound the space. By inspection, a greater simplification of the walls should help to determine the small set of basic instances derived from the generic wall that constitute the constructive vocabulary of designer's art of generating space.

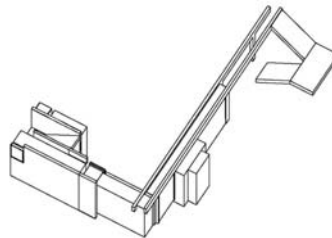


Figure 4-7: Example of spatial notation

The third level of notation, the diagrammatic level, foregrounds underlying, emergent boundaries of space and discards all connections between them. This level of notation, closely related to the parti of a design, the geometrical diagram or pattern that emerges when all details have been dropped out, is the most abstract version of the model and functions as a scaffolding of the design. Metric distances between boundaries are taken into account. Vitruvius defines ordering as the initial commitment to a geometrical system that controls the subsequent design, usually a modular layout (not necessarily a grid), because it consists of deciding the quantity of the module, and unites the individual parts to the overall proportional system.

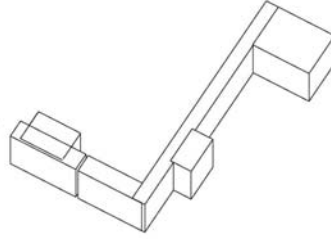


Figure 4-8: Example of diagrammatic notation

4.4.3. *Play mode*

The play mode is the application of group theoretical analysis in the simplest representation that has been found. All parts of the design that comply with the symmetry group of the configuration and all the symmetry subgroups are extracted and ordered in partial order lattices. The layered arrangement of all the parts the one upon the other should be able to give the complete simplest configuration that started the computation.

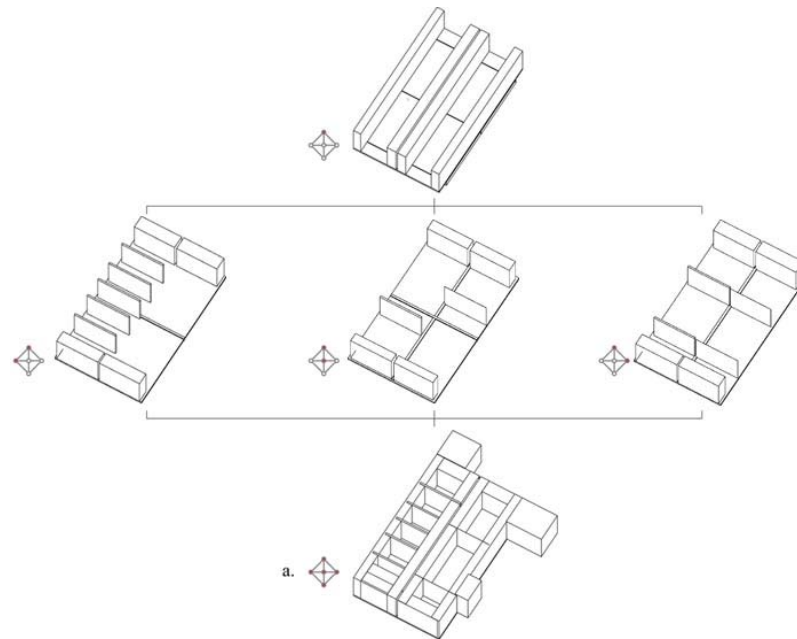


Figure 4-9: A partial order lattice of a simple arrangement

4.4.4. *Fast Forward mode*

The forward mode entails the reversal of the process from the simplest representation to the initial three-dimensional model. In this latter case the parts of the design that were parsed in the previous stage still come parsed in the successive layers and suggest correspondences that otherwise would be impossible to observe.

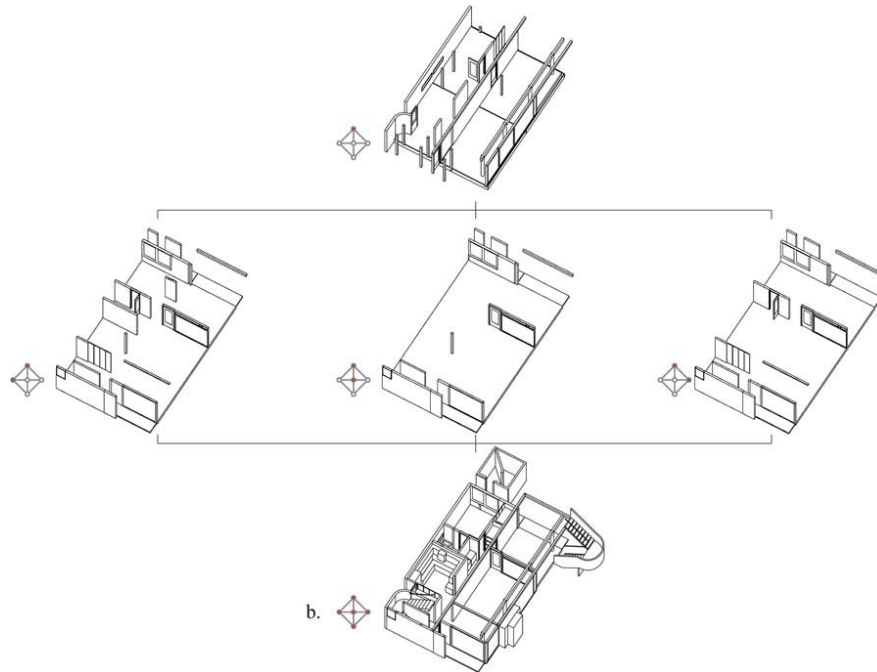


Figure 4-10: A partial order lattice of a complex arrangement

4.5. Summary

The basic tools necessary for a group theoretic analysis of architecture works typically characterized by terms such as asymmetry, complexity, weighting, projection, layering and so forth, have been outlined. The best way to demonstrate the applicability and usefulness of this analytical approach is to turn to actual implementation and analysis of designs of late modernism. The following chapter turns exactly to this goal and implements an in-depth analysis upon one of the exemplars of late modernism, the Smith house by Richard Meier.

Chapter 5 Visual computations

An application of this methodology in formal analysis is attempted here using Richard Meier's Smith House as a case study. All plans of the house are represented and decomposed in specific ways as described in the previous chapter and the computation of all symmetry parts takes place in entirely visual terms. A reassembly of the layered symmetries explains the structure of the symmetry of the house and provides an illustration of the basic thesis of this research on the foundation of a theory of emergence based on symmetry considerations.

5.1. Introduction

'Were architecture to be a dream of pure structure, Eisenman is the one who, more than any other in America, comes closest to achieving it; if, however, architecture is a "system of systems", if its expressions belong to different but interwoven areas of language, then it is Meier who is able to grasp those relationships.' Tafuri (1976)

The 1967 Exhibition New York Five (NY5) on the early work of five New York city architects, namely Peter Eisenman, Michael Graves, Charles Gwathmey, John Hejduk and Richard Meier, and the subsequent book Five Architects published in 1972, have indelibly stamped the course of the history of modern architecture of the late twentieth and early twenty-first century. The explicit reference of NY5 to the work of Le Corbusier in the 1920s and 1930s and its ironic allegiance to a pure form of architectural modernism made the exhibition pivotal for the evolution of architecture thought and language in the subsequent years and produced a critical benchmark against which other architecture theories of postmodernism, deconstructivism, neo-modernism and others have referred, critiqued or subverted (Tafuri 1976; Jencks 1990; Major 2001). Among this early work of NY5 the Meier's buildings were closer from all on the modernist aesthetic of the Corbusian form and in fact even the later buildings that Meier produced since then have all remained truest to this aesthetic.

The work here traces the history and logic of the evolution of Meier's early language and its direct relationships to spatial and formal investigations of early twentieth-century modernism as well as its direct reciprocal relationships with the rest of the NY5 languages. The departure for

this inquiry of such centrifugal relationships between rules and products and between notation and performance, for the purposes of this work is Richard Meier's Smith House, an early pivotal work and acknowledged forerunner and embodiment of the full repertory of Meier formal strategies and language (Colomina 2001).

The formal theory for the analysis of the language of the house is based on the model developed in the previous chapter. The specific methodology for performing this analysis relies on partial order lattice representations for the decomposition of a design (Park 2000; Economou 2001) (Economou 2001). Here this methodology is extended to describe complex spatial configurations characterized by architectural concepts such as layering transparency and collage (Slutzky 1989; Hildner 1997). More specifically, the analysis here uses all three levels of representations postulated by the model for the description of the house, each one specifically designed to bring forward different aspects of its spatial composition. The partial order lattice pictorially presents the symmetry structure of a rectangle-based spatial configuration. The number and qualities of the symmetry subgroups found in Meier's architectural composition provides the maximum number of layers. Analytically, these layers of the architectural design are used to reveal parts and sub-symmetries that are used strategically for the scaffolding of the design. Synthetically, group theory suggests operations and spatial transformations that may have been in compositional and thematic development of the design. A partial ordering of sub-symmetries and a classification by lattice diagrams of sub-symmetries exposes the underlying structure of the complex designs.

A major motivation of this work is that there is a correspondence between the evolution of architectural languages and the formalisms that can be used to describe, interpret and evaluate them. Classical modern buildings can be and have already been successfully described by group theoretical techniques. In the same way, Richard Meier's work constitutes a hyper-refinement of the modernist imagery that has been inspired not by machines but by other architecture that was inspired by machines and especially Le Corbusier (Goldberger 1999). Thus, the group formalism can describe Meier's architecture as a hyper-refined construction that relies on specific representations and mappings that foreground internal complex relationships of the structure itself, i.e. the symmetry subgroups and super-groups of any given spatial configuration. Here, all plans of the house are analyzed in terms of corresponding group structures and all are represented in partial order lattices using axonometric orthographic projections to illustrate notions of complexity, ambiguity and emergence and the ways they all inform design.

5.2. White geometries

'I would suggest that what distinguishes the Whites from Le Corbusier lies precisely in their elevation of form from the condition of design to that of epistemology. What was at stake was the claim that form was a type of knowledge; indeed, an essential type of knowledge.' Deamer (2001)

To understand the formulation of the formalist epistemology architecture takes under the New York Five, also known as the Whites, one needs to survey their legacy on one hand, and on the other, to understand the fundamental emphasis of the homogenous plane plays on their work on the surface. To understand this formulation, Deamer's (2001) account tells us that 'The true legacy of the Whites is not their formal vocabulary... but the fact that these (formal) operations have a systemic intellectual import at all.' The linking of form to knowledge is due to Hildebrand (1893), and later Arnheim (1954), but what is new is its import to late modernism.

The initial core of shared intellectual aspiration is formed in 1954-56 by the association of Colin Rowe, Robert Slutzky, and John Hejduk as "Texas Rangers" at Austin, and later Peter Eisenman with Colin Rowe at Cambridge in England (Caragonne 1995). While Rowe brings with him the legacy of Rudolf Wittkower who sees Renaissance as a scientific program, Slutzky as a painter brings the Arnheimian idea of depth on the picture frame, frontality as the dominant visual ordering system, and strong separations between foreground figure and background field. Graves, Hejduk, and Meier began their architectural search with their concern for organizing the visual world as in painting in a spatially complex manner. What the Whites have in common is to inject the most subtle commentaries on spatial layering, frontality, rotation, skews and ambiguity to the debate of the International Style. Some of their representative works are shown in Figure 5-1.

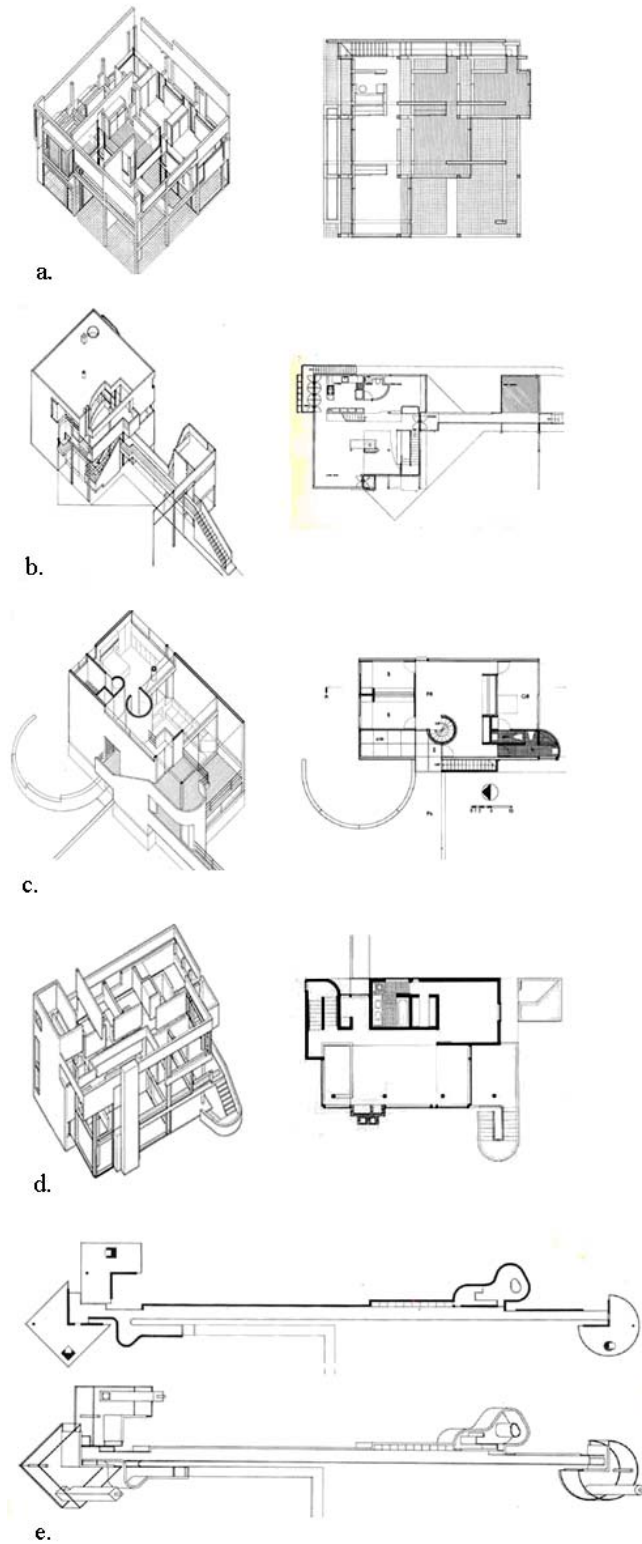


Figure 5-1: Representative work of NY5: a) P. Eisenman, House II; b) M. Graves, Hanselman House; c) C. Gwathmey, Cohn Residence; d) R. Meier, Smith House; e) J. Hejduk, House 10

Eisenman in House II in 1969 develops systematic analytic diagrams combining basic formal moves to sophisticated results (Eisenman 2004). Starting with a square volume marked by a matrix of sixteen square columns, its underlying structure is expanded by transformations based on sequential layering of solids and voids. Any element or relationship between elements has two notations, marks, or weightings of relative equivalence. He furthermore creates a sense of ambiguity between figure-ground, solid-void, window-wall though the definition of the elements as well as though their placement, size, and number (Figure 5-1a).

Graves in Hanselmann house in 1967 uses abstraction before migrating through collage to the poetics of neo-classical motifs (Graves 1982). The house is understood frontally by the layering of three principal façades. The first consists of a pipe rail frame and the front plane of a studio house. It acts as a gate, receiving the stair between the ground and the entrance level. The second primary façade, located at the center of the composition, contains the point of penetration into the main volume of the house. The third façade which is the densest is the real wall of the house's composition and the surrounding landscape. An outer terrace relates to the diagonal of the stream and implies a larger compositional frame (Figure 5-1b).

Gwathmey's abstract formal vocabulary in Cohn residence in 1967 is devoted to the interaction or interpenetration of contrasting platonic volumes and pure shapes (Gwathmey and Siegel 1984). Form relates to the line of interpretation between abstraction and representation. The work appears to rest in the Cubist frame of reference. Clearly, architecture is not skin-deed. To detach the bones from the skin is the beginning of formal irresolutions which deny the basic principles of the composite overlay of plan, section, façade that produce the building (Figure 5-1c).

Meier in the Smith house in 1967 uses geometry as a magnification of architectural functions with objects which display their function in absolute clarity (Meier 1984). The house has a layered structure, in which the relationships between volumetric order and transparency, and the analysis of possible geometric articulations suggest certain analogies to the structural purity of Eisenman and even to some ambiguous metaphors of Graves. Meier (1984) offers an architecture that presents itself as 'a system of systems' (Figure 5-1d).

Hejduk in House 10 in 1966 studies the formal propositions of the avant-garde to draw imaginary projects (Hejduk and Henderson 1988). 'To fabricate a house is to make an illusion'. In House 10, basic geometric forms (circle, square, diamond) are cut into quarters and are separated and grouped at the ends of a long path. By doing so, Hejduk performs two complementary tasks: he

chooses absolutely trivial forms, and uses the technique of volume deformation and that of sectioned volumes according to elementary rules (Figure 5-1e).

In all, clear compositional strategies to eliminate the vicissitudes of subjective seeing mark the work of the Whites: frontal-rotational, solid/void, grid/dissolution of the grid, virtual/actual solids and voids; whole and partial Platonic figures, regulating lines, datum, and golden proportions. This process of abstraction goes hand-in-hand with a concept of de-familiarization through the elements of the frame or the grid creating a Cartesian field. Thus, the function of the plane as a method of stratifying space implies a kind of wall whose primary function is to modulate space. The Whites share an enduring interest in Le Corbusier and in turn provoke the creation of an oppositional group, the Grays, who promote a less abstract architecture. Colin Rowe and Vincent Scully are supposed to be their respective backers. The difference between the Grays and the Whites is the supposed privileging of perception by the former and conception by the latter, while in fact 'the Whites have usurped perception to their own ends, making it a conceptual tool' (Deamer). But the operations that link them to Le Corbusier start with the grid dominated by field and figure that provide the framework for operations of transformations. What distinguishes the Whites from Le Corbusier, is namely their elevation of form from the condition of design to that of epistemology. As Deamer (2001) puts it, 'Le Corbusier never identified form in and of itself as the ends of architecture... One would never find in Le Corbusier, or any of the original modern architects, arguing, as Hejduk does, about the essential merit of the diamond over the square, or the necessity of revealing deep form in the environment, as Eisenman does... For the Whites, there is the relationship between percept(ion) and concept(ion) – if architecture is a form of thought, how does visual perception interface with that mental construct?'

The breakthrough toward this epistemological surge on formalism yields the reliance on forms of thought exterior to architecture, be they philosophical or scientific, half-materializing Lionel March's call for his adoption of scientific models for architecture. For Deamer, one of this phenomenological strain is the rise of Daniel Libeskind, Hejduk's former student, who combines Hejduk's poetry with Eisenman's conceptual logic. Greg Lynn, Eisenman's former student, developed his biological methods within the same framework. Needless to say that Cache's epistemology or Berkel's conceptual techniques are the offsprings of the theoretical territory prepared by the Whites (Deamer 2001).

The formal framework for understanding the underlying connection between the Whites and contemporary architects has also been facilitated by the rise of CAD systems in design that help link thought and images, perception and conception. It seems that the organizational scaffolding corresponding to the way the Whites link the forms of architecture to the layering is so algorithmic that their generative reliance on numerical predictability attracts contemporary designers to inherit the Whites' design methods via computer versatility (Deamer).

This move is also facilitated by the current trend to emphasize work on the surface as the primary conceptual device – it is the surface Libeskind writes on, the surface Lynn folds, the surface the computer turns into topography for Cache. The planes deployed by the Whites are the phenomenal datum onto which three-dimensional spaces collapse. The planes deployed by the digital architects mostly are not. One can read the most recent surface work as an emancipation of painterly two-dimensional surface from the no longer dominant grid and volume. That kind of work is not a-spatial per se, but the complex organizations of physical matter yield complex spatial interiors. Although the spaces are never conceived of positively, the common trait remains: complexity and ambiguity. The Finnish architects conclusively make a comment on the added meaning of the simplified box that gives the spectator a new variation and a wider possibility of viewing the third dimension more completely. The combined effect of layers of different elements and materials creates a new kind of homogeneity. What Reima Pietila explains: “These model house developments of the sixties and seventies have quite another feeling for form compared with Le Corbusier’s twenties: that of cold computer intelligence... The Futurist present of that great mystic Le Corbusier has come to nothing and has been replaced by the new metamorphoses of continued space and the infinite permutations of the sixties.” (Stenros 1987)

5.3. The Smith House

‘Meier’s hyper-refinement of the modernist imagery has been inspired not by machines but by other architecture that was inspired by machines... honoring his “fathers” and casting them off at the same time.’ Goldberger (1999)

It is clear that the complexity suggested in the reading of the corpus of the NY5 would make any of the buildings belonging in the set an ideal candidate for a case study for the analytical method

developed here. Still, it is argued here that among all possible candidates, the Smith House stands out as the right candidate. The house has a long legacy: Frampton (1975) and Johnson (1975) nominated the Smith House as a classic and selected the young Meier as the one architect out of five who knows history the most and learns from it; Rykwert (1991) has asserted that the house is a classic case of one that has been designed upon a formal vocabulary whose elements are all abstracted from the repertory of early modernism and juxtaposed back as a collage; Jencks (1990) has asserted that Meier uses a mixture of traditional forms of modernist architecture and where the system is incomplete, new elements are added. And still many other key discourses have been suggested to include the themes of compositional grid and patterned frames (Goldberger 1999), the discipline of the Dom-ino and Citrohan structures (Kupper 1977), Mies' aesthetic of rhythmic linear elements (Hildner 1997), and several others. It is clear that Meier's language, iconography, and elemental categories force comparison and differentiation with the work of the other members. Meier's long standing personal affiliations with artists and his torn-paper collages are his technical link to the painterly means of space-making by the use of color, surface, line, and contour.

For the purposes of this research, and in addition to what has been mentioned so forth about the house, a key aspect of the house is that it embodies the quintessential aspects of the abstract modernist vocabulary in that it exemplifies the organization of space through the abstract instruments of plan and section. The Smith house is the first of Richard Meier's white buildings, which the architect characterizes as: "the precise manipulation of geometry in light that translates into power of architecture to become art" (Meier 1996). Most importantly, it is the project that exemplifies the most the wall not as an element of construction but as an abstraction, as a spatial element, a homogeneous plane [Meier in (Davies 1988)]. The architect considers this house his first mature building: "It was there ... that I was first able to develop and test a number of issues that I had been preoccupied with. In fact, those issues still preoccupy me: the making of space, the distinction between interior and exterior space, the play of light and shadow, the different ways in which a building exists in the natural or urban world; the separation of public and private space" (Meier 2000). All analytical drawings that will illustrate the formal analysis here are based on high resolution copies from 'Richard Meier Architect' Volume 1 (1984); no other primary or secondary evidence about the drawings of the house are used except the copies of the drawings published in the book.

5.3.1. A first encounter: Site Structuring

The Smith House is in Darren, Connecticut, and it is situated on a 1.5 acre site overlooking Long Island Sound from the Connecticut coast. The house was built during 1965-67 on a site literally adjacent to the water and it was designed for a family with two children. The site plan of the house is shown in Figure 5-2.

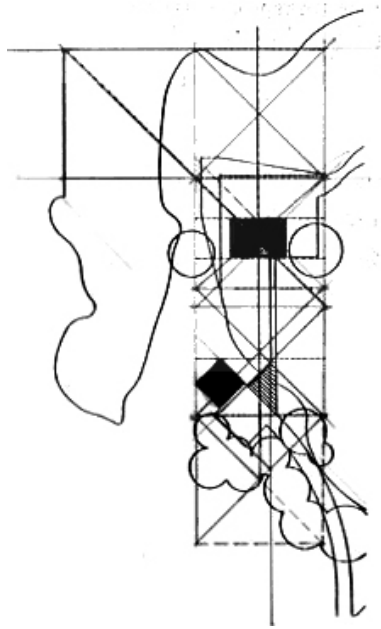


Figure 5-2: Site plan of the Smith House

The house is developed over three levels. The entrance area and master bedroom are on the middle floor. The lower level is for dining, kitchen, laundry and domestic help. Both the living and dining areas open directly to outdoor terraces. The top floor contains children's bedrooms, guest-room and library-play. The house is finally topped by an outdoor roof deck. The three plans of the house are shown in Figure 5-3.

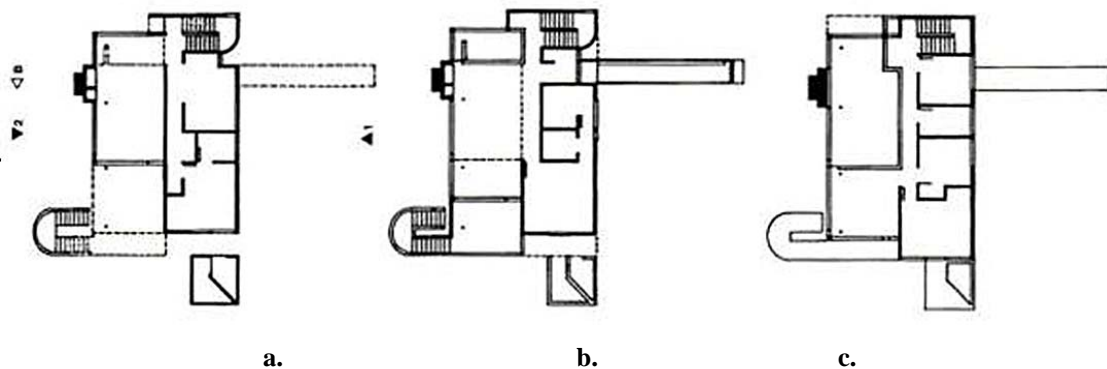


Figure 5-3: Plans of the Smith House. a) Lower floor; b) Middle floor; c) Upper floor

The house itself appears to be a hyphenation of two canonical structures: the Citrohan house and the Domino house (Corbusier and Jeanneret 1937). The Citrohan zone is a series of closed cellular spaces and the Dom-ino zone is leveled as three platforms within a single volume enclosed by a glass skin. Meier investigates a language of oppositions of a denied dialectic between the total transparency of the panoramic façade and the solid compartment of the entrance façade. The handling of the layer stratification of the building parallels the post-Cubist conception of spatial relationships. On this basis, the conception of the spatial arrangement of the house too parallels the development of combinations and assemblages of lines, planes and volumes, independent of what the given elements may represent. Two facades and sections of the Smith house are shown in Figure 5-4.

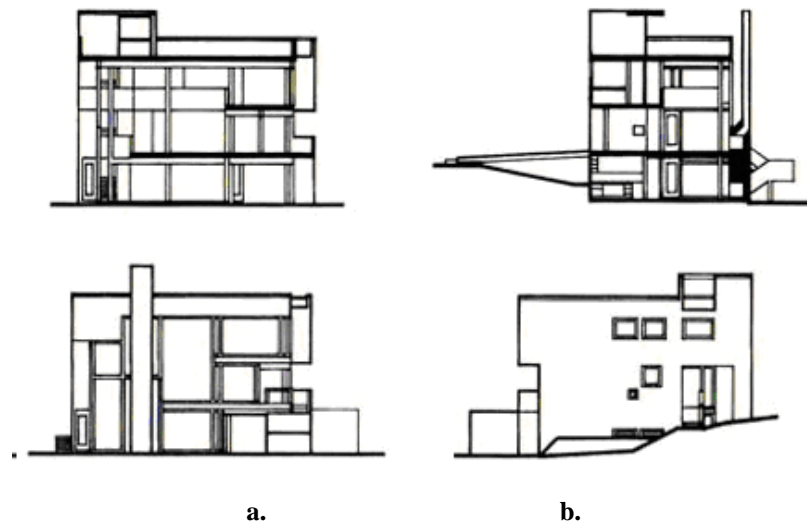


Figure 5-4: Orthographic views of the Smith house. a) Longitudinal sections; b) Transversal sections

5.3.2. *Second encounter: Maximal lines*

‘... two different classes of proportion, both derived from the Pythagorean Platonic world of ideas, were used during the long history of European art... The Middle Ages favoured Pythagorean-Platonic geometry, while the Renaissance and Classical periods preferred the numerical, i.e., the arithmetical side of that tradition.’

Wittkower (1978)

A typical framework for formal analysis is the identification of all possible regulating lines in plans and facades and the examination of the characteristics of the emergent shapes and configurations of the regulating lines. Typically such an analysis, especially for rectangular geometries, proceeds along extensions of the walls to provide grids and shapes with special characteristics, say squares, root-two rectangles, golden-section rectangles and so forth as well as spatial relationships between them, say, center to center, center or edge, edge-to-edge, and so forth. Several variations of these regulating frameworks are given in Figure 5-5.

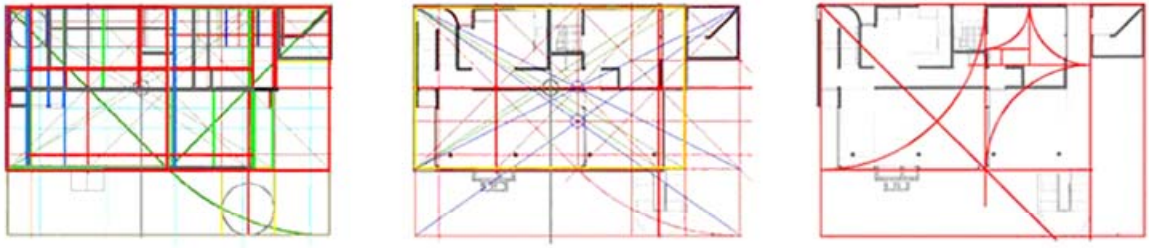


Figure 5-5: Search for alternative partitions. a) Root-2 lines; b) Candidate centers; c) Golden Ratios

All regulating lines provide several alternative partitionings of the house and they are clearly based on the geometry of the rectangle. These rectangles come up in various sizes and dimensions depending on what is subsumed under them. Still, it is clear that all those can be grouped in three general classes of arrangements depending on the choice of center for the selected rectangular configuration. Possible decompositions of the house include: a) a major rectangular space with three secondary spaces attached to it in right angles relationships; b) a major frontal rectangular space that is interspersed by the element of the staircase; c) or alternatively a major rectangular space that subsumes all parts of the house and all spaces are thought of as carved out from the major body of the house. Other readings are certainly possible. All these decompositions suggest alternative solutions to the compositional problem of identifying, if there is, a common appropriate center and axes of symmetry and disposition of the configurations. The three cases explored above are given in Figure 5-6 as variations exploring the minima and maxima of bounding rectangles. The parti of this house is based on the first case called minimum rectangle.

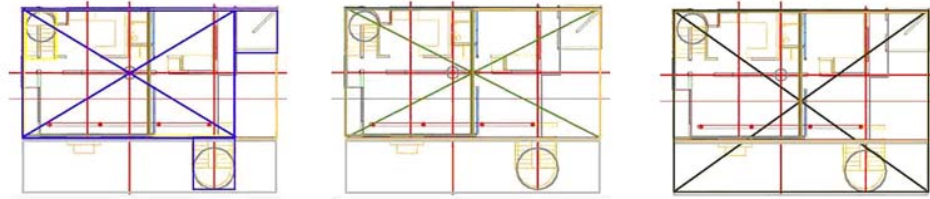


Figure 5-6: Three decompositions of the house in terms of a major rectangle a) Minimum; b) Medium; c) Maximum

The same regulating lines as extensions of walls and correspondences between them, construct as well systems of lines and emergent grids that are used to interpret space according to a pattern of oppositions: vertical/horizontal, top/bottom, orthogonal/diagonal, left/right. These grids can be drafted on various directions foregrounding the x direction or the y or even diagonal relationships between the cells. Three superimposed grid systems for the three levels of the house are shown in Figure 5-7.

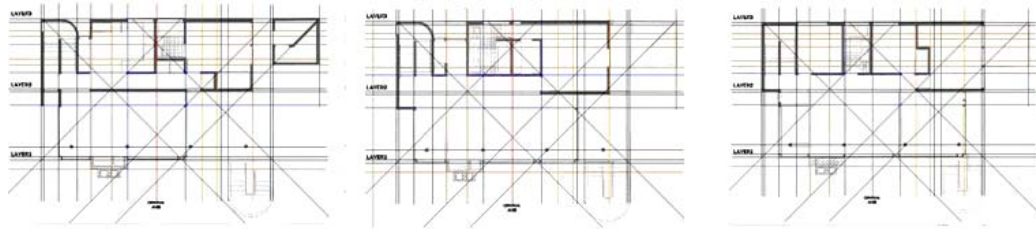


Figure 5-7: Superimposition of grids for the three levels of the house

These grids show their ability to be used as organizations devices for the interpretation of space when all walls are eliminated and the regulating lines are shown by themselves. These scaffoldings compose then the underlying organizations structure of the house and foreground arithmetical and commensurable relationships between the parallel lines. The three grid systems extracted from the wall extensions for the three levels of the house are shown in Figure 5-8.

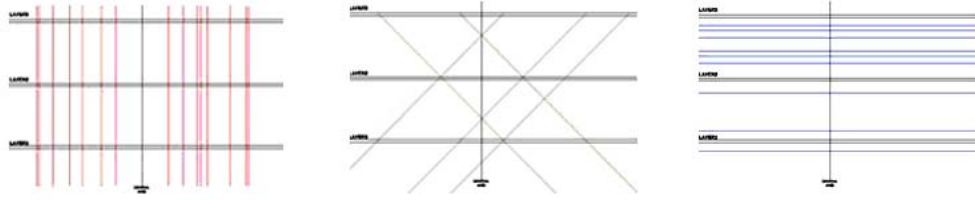


Figure 5-8: Rhythmic dispositions of grid

5.3.3. *Third encounter: Planes and walls*

'I am interested in spatial elements, not elements of construction. I want the wall to be a homogeneous plane.' Meier in Davies (1988)

Here a third encounter with the description of the house is suggested and this time the wall element is foregrounded. Meier himself has attested to his preference to spatial elements rather than construction elements and especially his predilection for the wall to be a homogeneous plane. A close examination of the instances of the wall in the house and their spatial relations suggests compositional processes such as parameterization, dematerialization, deformation, defragmentation and therefore point to the design of an additional overall framework for a critical description and interpretation of the house. This suggestion here is based on a series of experiments upon the representational elements of the house and their consistent typological reduction in the planar unit of the wall. This is not as easy as it sounds because the geometric rectangular prisms of the house resist their immediate interpretation: Often they appear as massive blocks - space volumes, other as opaque walls with or without openings; and lastly as emergent planar shapes that organize space. Different interpretations of these rectangular prisms as walls of different kinds are shown in Figure 5-9.

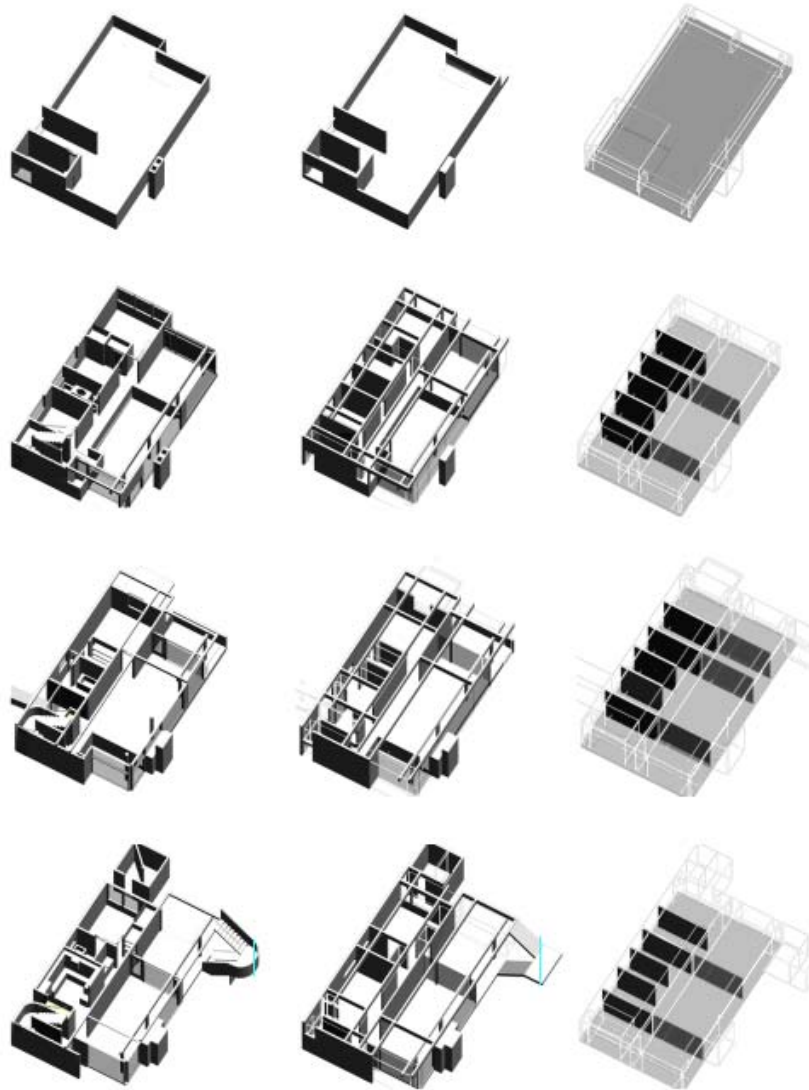


Figure 5-9: Different interpretations of rectangular prisms as space volumes, perforated walls, or compositional planar partis.

The principle applied at the elevations and sections of the house suggests somewhat similar and different interpretations. The most immediate finding is that the same root-2 rectangles that can be found in the plan can also be found in the elevations and the sections. That is not surprising given the abstract modernist vocabulary of the house and its exemplification of the organization of space through the abstract instruments of plan and section. What is more interesting is that the same trivalent condition (T-shape formation) of the intersection of the lines in the plan exist in the façade too but now it is even more celebrated in various ways constructing essentially grids in essential nested ways. The rules that can account for such a T-shape intersection are straightforward as shown in Figure 5-10.

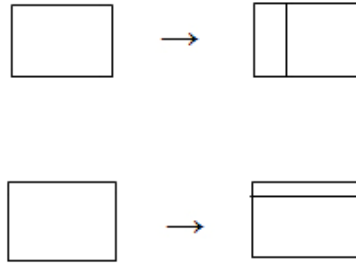


Figure 5-10: Rectangular divisions of the rectangle

Successive applications of such rules produce quintessential modernist arrangement with nested grids populated by T-Shape intersections. A derivation of a typical nested T-shape grid is shown in Figure 5-11 and a series of T-shape intersections found in the façade and the section of the Smith house are shown in Figure 5-12.

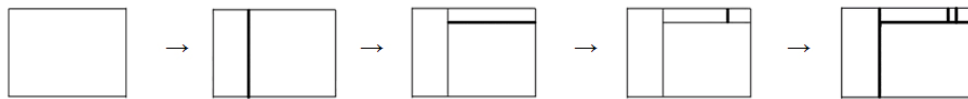


Figure 5-11: A derivation of a nested T-shape grid in the Smith house

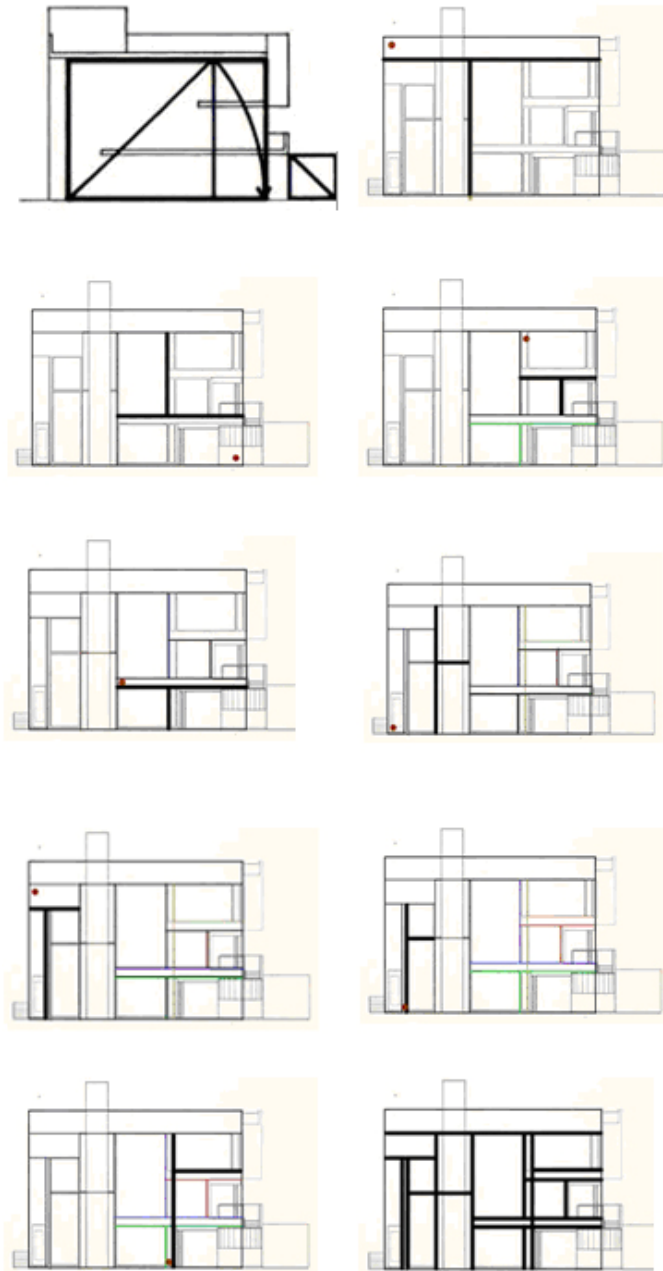


Figure 5-12: Rhythmic dispositions of grid

The basic topology of the wall can be seen as a planar rectangular element with cutout parts to account for openings of all sorts, including doors, windows, and thresholds of several conditions. The planar element can be modeled after a dimensionless gridiron pattern whose cells may denote closed and open parts in the wall and the formal representation of the wall is thus taken as a binary configuration based on an $n \times k$ grid. Among all possible gridiron systems the 3×3 was

chosen here as the most generous for architectural purposes. The basic gridiron pattern of the wall is shown in Figure 5-13.

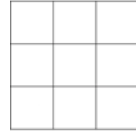


Figure 5-13: The cycles of permutations for the 3×3 cell

The task is to identify all possible non-equivalent combinations of closed and open cells that can be found in this configuration and map these structures to the set of walls that Meier uses. In other words, what is suggested here is that the specific decomposition of the three-dimensional cuboid in Meier's work can be seen as based on a decomposition of the two-dimensional square into specific gridiron systems and that by de-fragmenting the modules of vertical planes to determine the classes of openings in a planes, Meier's palette can be easily put on display.

The method of counting of non-equivalent configurations based on a given permutation group of vertices of a geometric shape has been given by Polya in his theorem of counting. The description of the formalism and various applications in other domains has given elsewhere [(Polya and Tarjan 1983); (Economou 1999); (Din and Economou 2007)]. Here only the application of the theorem on this specific context is given. The most critical part of the application is to identify the model of the structure. The 3×3 grid is here taken as a 3×3 cellular structure whose cells are represented by vertices. Polya's theorem will then provide the answers about the binary condition of these vertices. Figure 5-14 shows the remodeling of the nine lines into nine vertices.



Figure 5-14: The 3×3 grid represented as an array of 3×3 cells

The core of the theorem is that any shape can be represented as a function of the cycles of permutations of vertices f_r that here are induced under the symmetry group of the shape, here the D_4 , the symmetry group of the square. Figure 5-15 shows all the eight cycles of permutations induced by the eight symmetry operations of the square. It is worth noting that the vertical, horizontal and diagonal symmetries induce the exact same permutation of the vertices and different from the half-turn symmetries; this is quite different behavior from the permutations of

the vertices of the square under the same operations where the half-turns induce the same permutations as the vertical and horizontal but not the diagonal.

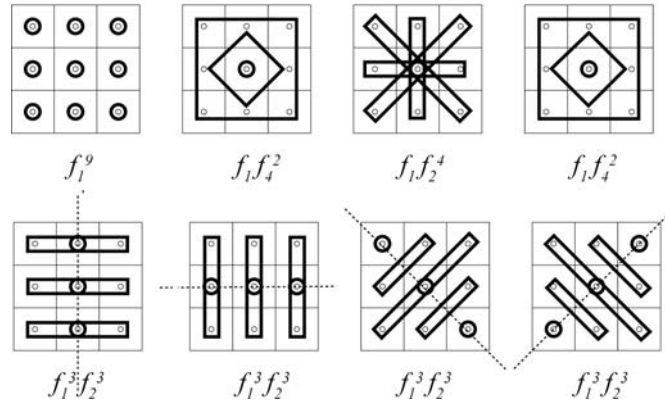


Figure 5-15: The cycles of permutations for the 3x3 cell

The sum of all the cycles of permutations and their products divided to the sum of permutations of the symmetry group of the figure provides the cycle index of the figure, the blueprint for the enumeration of all the possible subsets. Here, for a figure inventory $x+y$ where x and y represent the quantities that will be enumerated – the closeness and openness of the cells, its expansion according to the theorem is given in (3).

$$f^r = x^r + y^r \quad (3)$$

If by substitution the figure inventory into the cycle index replaces the cycle f_r with $x^r + y^r$, and expands the cycle index in powers of x and y , the resulting coefficient of $x^r y^s$ is the number of distinct ways of configuring the x cells and y cells with respect to the permutation group. The equation can be solved in a straightforward way by taking advantage of the multinomial theorem in (4).

$$(x+y)^n = \sum_{r+s=n} \frac{n!}{r!s!} x^r y^s \quad (4)$$

In the specific case here for the 9-cell grid the computation of the equations (3) and (4) for the figure inventory given in Figure 7 provides the complete set of solutions and that is a total of 102 distinct configurations. These configurations are symmetric regarding the quantities x and y . The 102 n -cell configurations for $x+y \leq 9$ are shown in Figure 5-16.

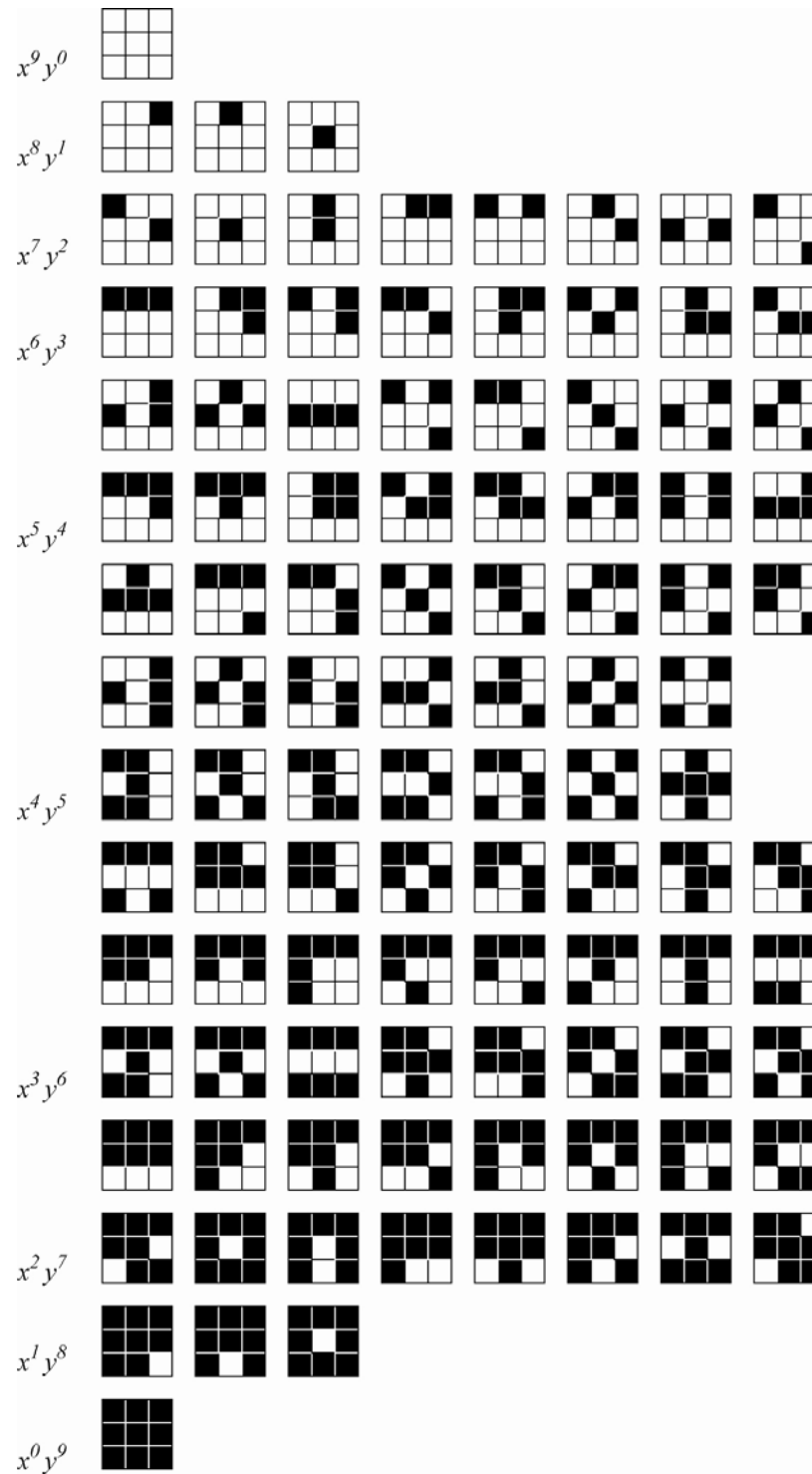


Figure 5-16: The 102 n-cell configurations of a 9 square-grid for x white and y black cells

The exciting part of this enumeration is that it provides the complete set of all possible configurations of all binary systems embedded upon a given grid and therefore it provides a systematic framework to explore all the possibilities implicit in the system. It is clear for

example, that some of these configurations have been used in many different circumstances in the design of the Smith house; these configurations consist of arrangements of black and white cells that denote respectively open and closed spaces or some hybrid in-between spaces. All walls are then understood as abstract geometrical cuboids exemplifying these abstract configurations as shown in Figure 5-17.

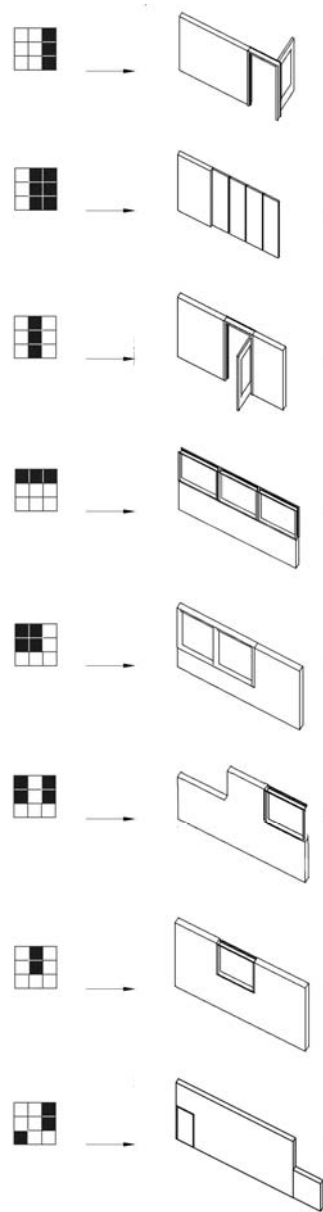


Figure 5-17: Non-equivalent configurations of a 9 square-grid in Meier's planar units

The parameterization of the block produces variations in dimensions, density and edge condition. Interesting cases emerge: A massive piece of wall or block can generate any of the most unlike elements: chimney, closet, recess, threshold, staircase, and so on by subtractive operations. A

solid opaque wall can be subject to operations of filtration, permeability or translucence. A solid transparent wall may instantiate either a glazed curtain wall or a window wall. Finally, a virtual wall as an abstract plane is then defined by its edge condition. A line on the plan may mark the separation of inside–outside, but it can also signify the edge of the volume, a change in material or level, or the presence of something above or beyond. A combination of a solid - opaque wall and a transparent glazed wall may yield a translucent wall. A combination of a solid wall and a space volume yields to a hybrid wall and so forth.

The exterior walls of the house can be nicely captured with these definitions. The frontal wall of the house is a triplet of planar and volumetric elements imbued with materiality and permeability. The glazing element incorporates open frames of wood| steel with an infill of glass. The trabeated element plays the role of concentric shell which acts as another filter. In-between is found an appended volume of space. The lateral window facade is layered same as the frontal one. Hybrid units are layered in parallel. Here, all the enclosing walls are hybrid. In the middle, there is the medial wall to create a vertical layer for the promenade architectural and to structure deep and shallow space. The south frontal wall of the house is a triplet of planar and volumetric elements imbued with materiality and permeability and generated by a block. The glazing element incorporates open frames with infills of glass.

The basic unit of the composition of the Smith house, the wall, with all its variations can be seen a vocabulary that comprises a subset of a specific set of topological transformations of rectangular prisms and correspondingly of the full vocabulary of the NY5 architecture.

5.3.4. Fourth encounter: Layers

According to Birkhoff(1933) the rectangular forms are best suited for use in composition. This is an algorithm to construct the five subgroups of symmetry of the rectangle that will populate the lattice. It will be made clear how they derive from one another, how some idea of weight or precedence can be set, and how object-feature mapping is applied as a function after Ho (1982 d). The search of the possible partitions of the rectangular frame of the building leads to multiple choice configurations. An emphasis on the golden section rectangle is considered here as a canvas. This kind of rectangle seems the most appropriate among the other choices, and it supports most of the axes of symmetry. The following steps report the result of this investigation.

Meier himself hints the starting block by providing the six partitions of the Citrohan block ($\sqrt{2}$ rectangle). The first function $f_1: V \xrightarrow{f_1} V$, maps of vertical reflection set into itself, with a restriction upon the range of (V, f_1) to the bottom half-plane so that,

$$f_1(v) = |v| \text{ such that if } v > 0, f_1(v) = -v, \text{ and if } v < 0, f_1(v) = v \quad (5)$$

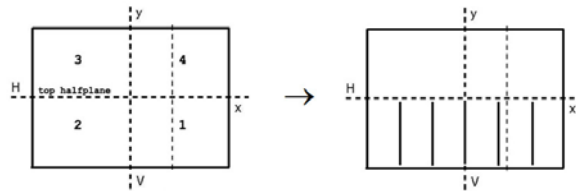


Figure 5-18: Vertical reflection (V)

The second function $f_2: V \xrightarrow{f_2} H$, maps horizontal reflection from the set V based on the Citrohan part into the top half-plane sheltering the Domino part. Thus, another restriction sets the range of (H, f_2) to the top halfplane as a well-defined mapping. For any h in H there exists v in V (onto function) such that,

$$f_2(v) = h \quad (6)$$

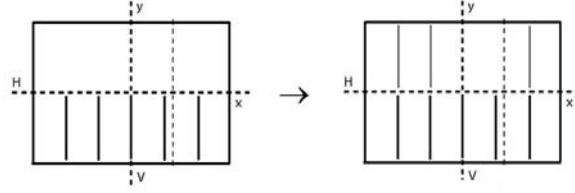


Figure 5-19: Horizontal reflection (H)

The third function $f_3: S_H \xrightarrow{f^2} S_V$, maps rotation or half-turn or spin from the set H belonging to Domino part into (top half-plane) into corresponding elements in the Citrohan part.

$$f_3(s_h) = s_v \text{ is a rotation through angle } \theta \ (\theta=\pi/2) \text{ such that } \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (7)$$

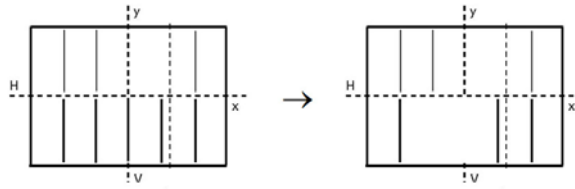


Figure 5-20: Half-turn (S)

The fourth function f_4 maps the three horizontal layers corresponding to the layered frontal, walls.

$$f_4: D_2 = \bigcup (V, H) \xrightarrow{f^2} D_2 = \bigcup (V, H) \quad (8)$$

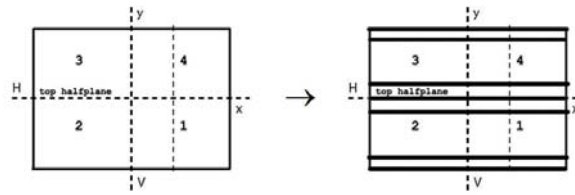


Figure 5-21: Dihedral D (combined V, H, S)

One subgroup is a placeholder for the elements that are unique in the house. These are add-ons labeled C_1 . Call them α -singularities, or α -unlike elements. In the Smith house there are five of them: chimney, interior staircase, exterior staircase, ramp, and cottage.

$$f_5(v) = \alpha \quad (9)$$

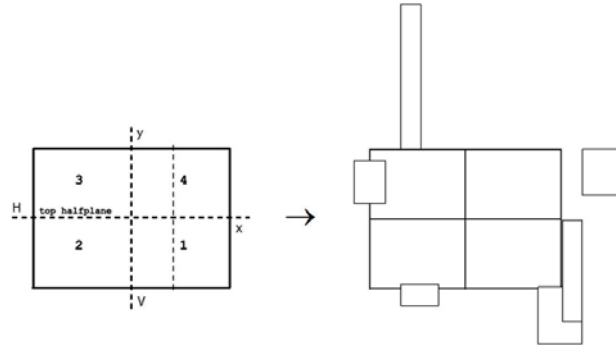


Figure 5-22: Singularities (C)

5.4. The Smith House: A formal description

The analysis proposed here proceeds along visual computations that are all based upon the models of analysis presented so far and uses representations that capture some, but of course not all, conventions characterizing a design. The key idea behind these computations is that they are designed to decompose the house in sets of basic elements that are then recomposed to re-describe the house and help interpret the basic assumptions about the system itself. The four aspects of representation used are abstraction, projection, weights, and layering. All definitions have already been given; here only a brief précis is given again: Abstraction captures the geometrical characteristics of the architecture object at different levels of detail. Projection denotes the specific orthographical or oblique mapping of the model of the architecture object upon the plane of depiction. Weight denotes the physical characteristics of the architectural object such as opacity, translucency, or transparency and is captured by three types of lines used here: Solid, thin, and dotted. Layering denotes the decomposition of the design in distinct parts. Other aspects of architecture representation routinely used in architecture notation are omitted here.

5.4.1. *Initial shape*

The initial shape that starts the computation is the three-dimensional model of the Smith house that is built upon existing plans, elevations and sections of the house. An axonometric view of the three-dimensional model of the house is given above in Figure 5-23.

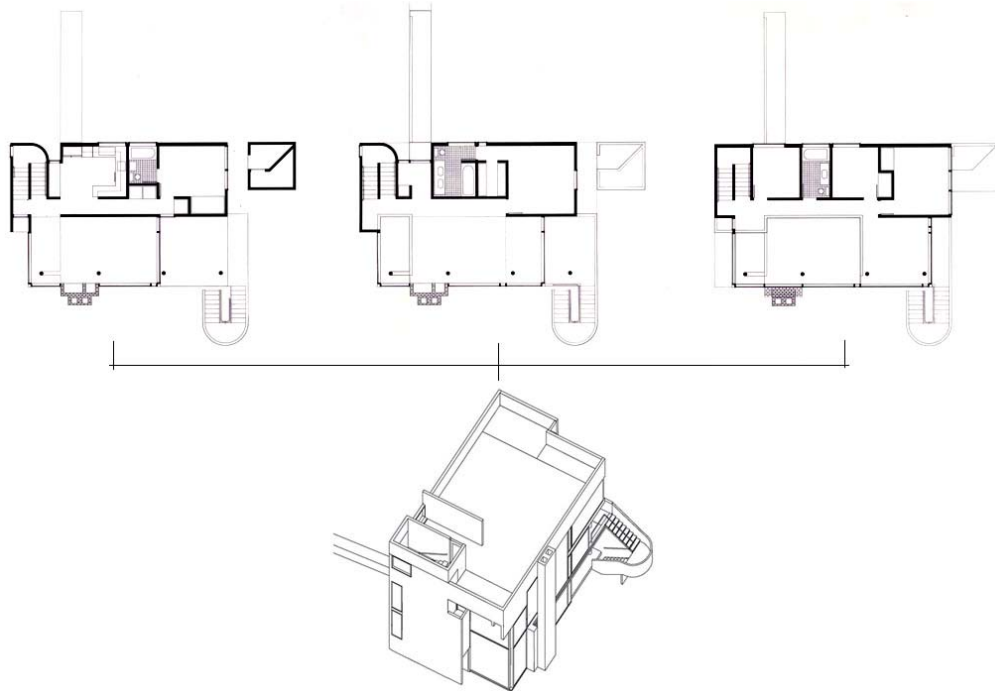


Figure 5-23: Axonometric view of the Smith House

5.4.2. *Rewind*

Three levels of abstraction are considered here as generous enough to capture key stages in the description and interpretation of the design. These levels are organized respectively as notational languages that describe some but not all features of the architecture space.

The first level of notation, code-named here as ‘architectonic level’, is the level that approximates in some way the original notation used for the language of the Smith House and for the most part the rest of the NY5 designs. The notation privileges functional elements such as walls, slabs, columns, and beams, walled furniture, handrails, and openings of various kinds such as windows,

doors, stairwells, chimneys, and so forth. The architectonic representation of the Smith House for all three floors is shown in Figure 5-24.

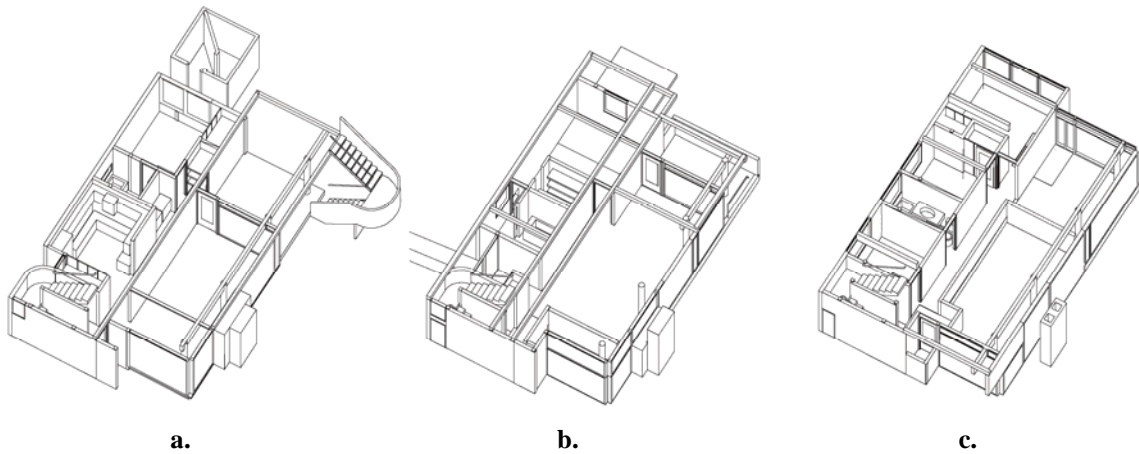


Figure 5-24: Architectonic level: a) Lower floor; b) Middle floor; c) Upper floor.

The second level of notation, code-named here as ‘spatial level’, privileges space divisions and corresponding openings in these boundaries, and discards all other information. This level essentially picks up planes that function as walls and slabs and so on, the connections between them and their interface with context for ventilation, light and so forth. The spatial representation for all three floors is shown in Figure 5-25.

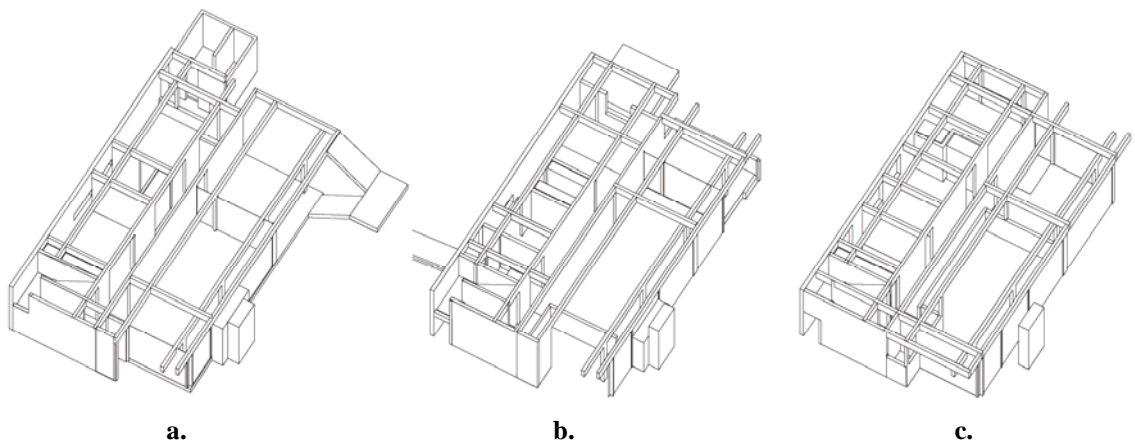


Figure 5-25: Spatial level: a) Lower floor; b) Middle floor; c) Upper floor.

The third level of notation, code-named here as ‘diagrammatic level’, foregrounds underlying, emergent boundaries of space and discards all connections between them. This level of notation, closely related to the parti of a design, the geometrical diagram or pattern that emerges when all

details have been dropped out, is the most abstract version of the model and functions as a scaffolding of the design. Metric distances between boundaries are taken into account. The diagrammatic representation for all three floors is shown below in Figure 5-26.

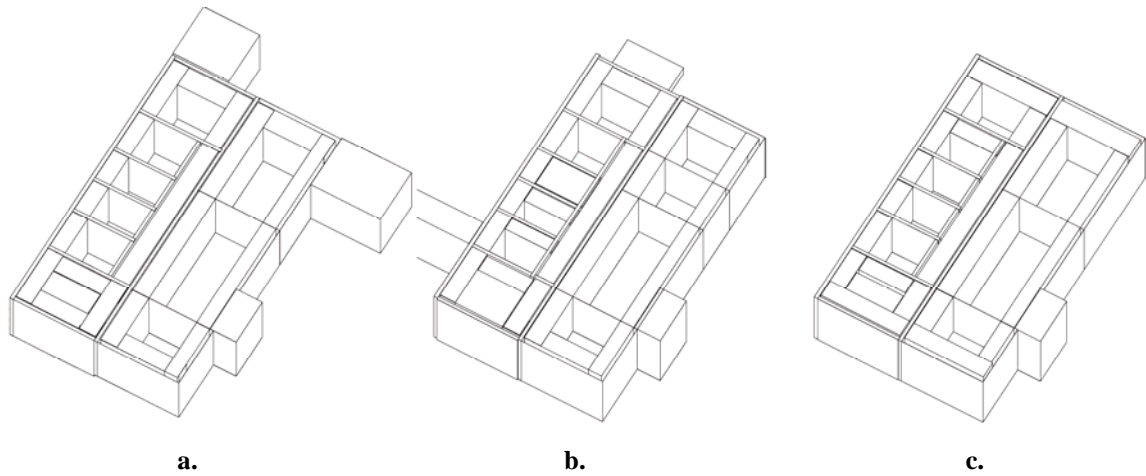


Figure 5-26: Diagrammatic level: a) Lower floor; b) Middle floor; c) Upper floor.

A closer look at the parsing of the model shows the application of a set of shape rules that replace parts of the model with corresponding parts in the level below them. Some shape rules are quite straightforward and they apply on simple continuous rectangular prisms with one or more undulations on their boundaries or openings within them to denote openings. In most of these cases the topology of the rectangular prisms is genus-0 (no interior holes) or genus-1 (one interior hole) (Banchoff 1990), typically associated with a window for a vertical rectangular prism denoting a wall or a staircase for a horizontal rectangular prism to denote a floor plate. A sample of these rules for the three levels of representations of this model is shown in Figure 5.27.

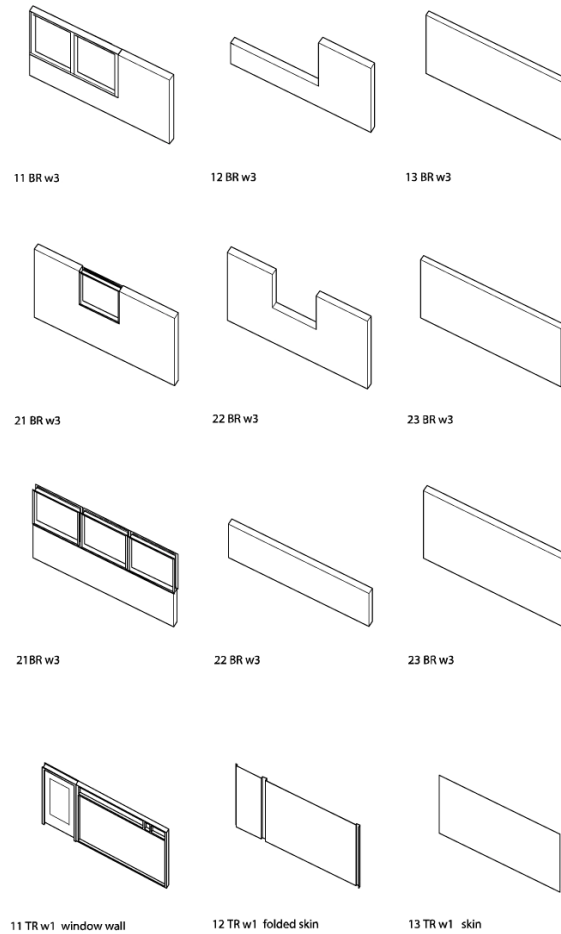


Figure 5-27: Successive abstractions of simple wall elements. Left column: Architectonic level; Middle column: Spatial level; Right column: Diagrammatic column

The simple rules shown above can be combined with one another to describe more complex spaces bounded by rectangular prisms. Often the interpretations and the parsing of the design are considerably harder than that and the straightforward application of simple boundary conditions cannot capture some of the subtleties of the design. In these cases the rules are more complex too and refer primarily to dihedral space conditions where the faces of the three-dimensional shapes turn to bound convex space. These latter cases involve decompositions of complex architectural arrangements in sets of maximum numbers of large convex spaces (Peponis and al 1997). A sample of these complex rules for the three levels of representations of this model is shown in Figure 5-28.

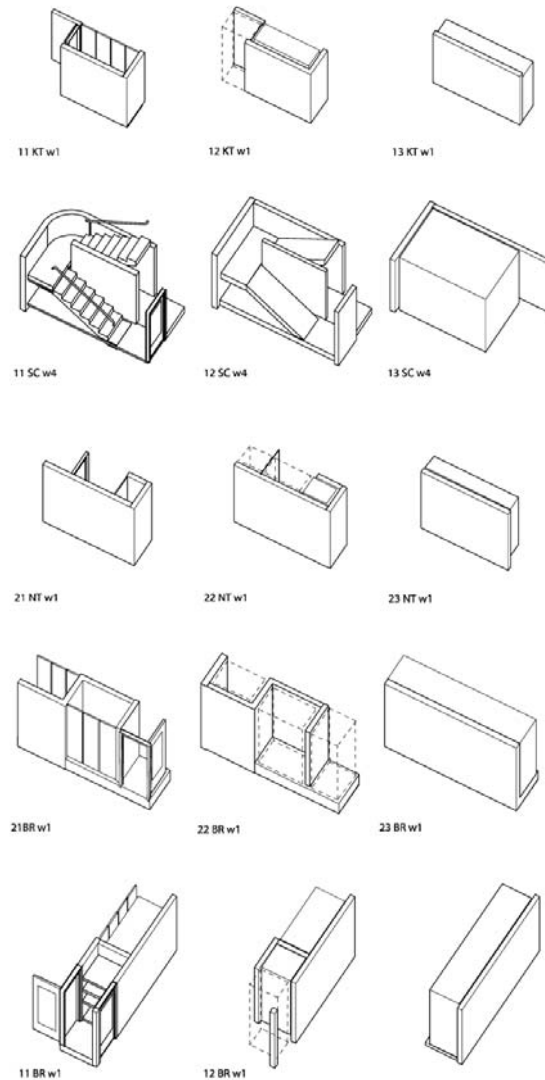


Figure 5-28: Successive abstractions of complex wall elements. Left column: Architectonic level; Middle column: Spatial level; Right column: Diagrammatic column

The realignment of the plans of the house in terms of successive levels of abstraction suggests an alternative reading for each floor that open up the issue of complexity to simplicity in terms of subtracted information from the drawings. The previous reading of the house in terms of parallel representations and computations, that is, all levels of the house given in a singular manner, say, architectonic, spatial or diagrammatic, aspired to a coherent totality of a representational mode. The realignment of these representations in terms of spatial indexing privileges now a comparative contextualization of the house. Furthermore this relationship establishes straightforward numerical relationships between the numbers of objects modeled in the

corresponding three-dimensional models of the house. For example, the number of objects depicted for the first floor of the Smith House is eighty, forty-nine, and twenty eight for the architectonic, spatial and diagrammatic level respectively and are shown in Figure 5-29.

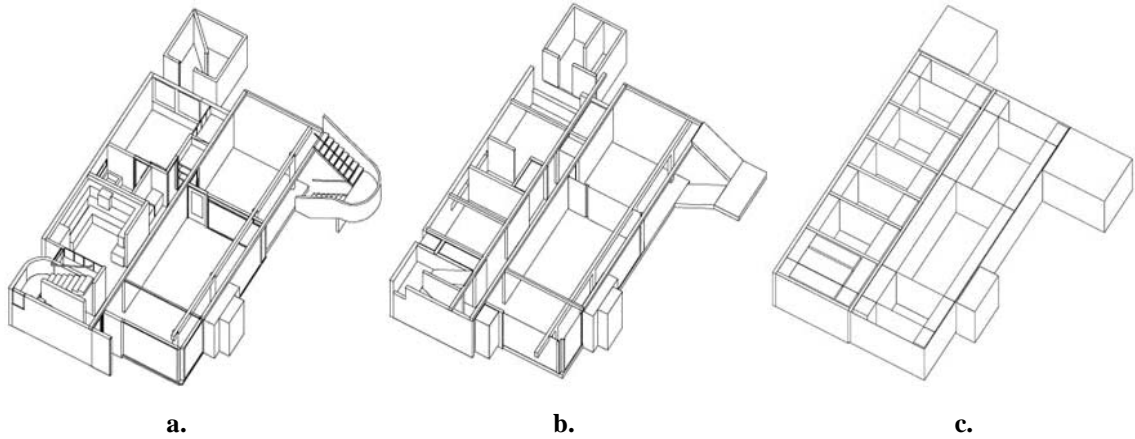


Figure 5-29: Smith House first floor - notations: a) Architectonic; b) Spatial; c) Diagrammatic.

The number of objects depicted for the second floor of the Smith House is seventy eight, fifty, and thirty four for the architectonic, spatial and diagrammatic level respectively and are shown in the Smith House Figure 5-30.

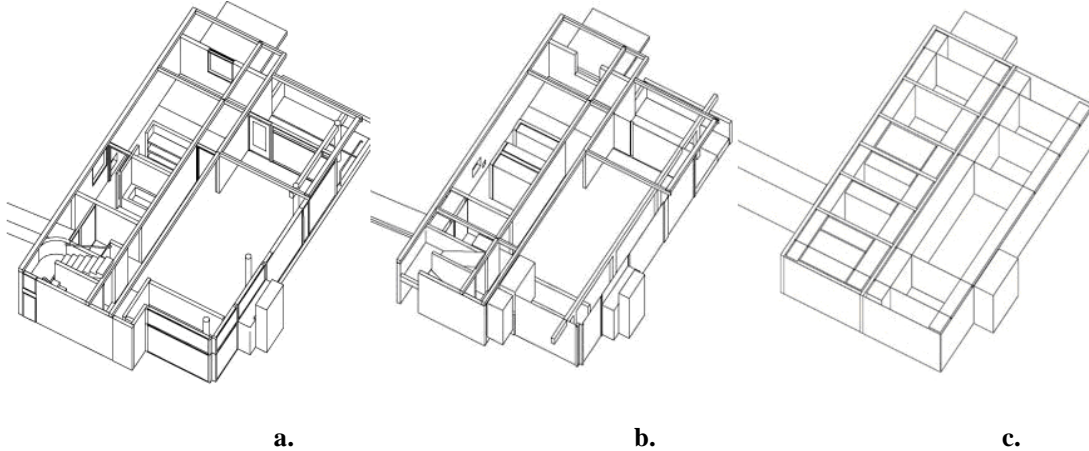
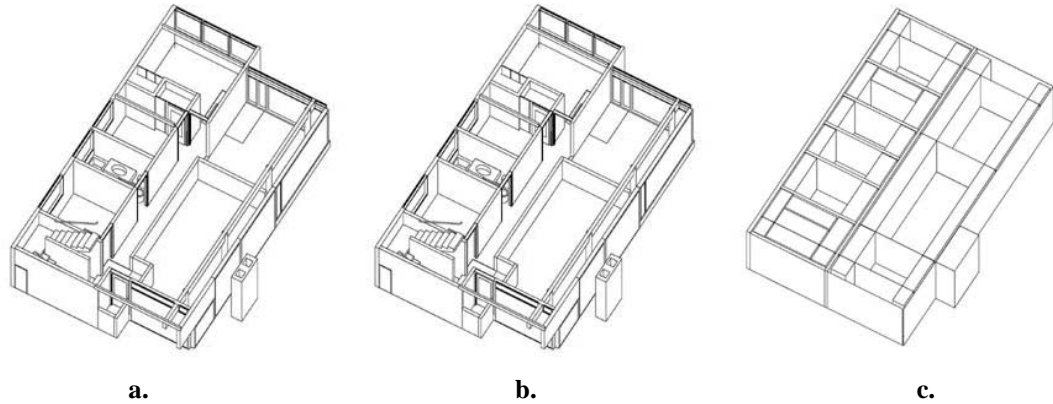


Figure 5-30: Smith House second floor - notations: a) Architectonic; b) Spatial; c) Diagrammatic.

The number of objects depicted for the third floor of the Smith House is seventy two, thirty six, and twenty eight for the architectonic, spatial and diagrammatic level respectively and are shown in the Smith House Figure 5-32.



Smith House Figure 5-31: Smith House third floor - notations: a) Architectonic; b) Spatial; c) Diagrammatic.

Lastly, the number of objects depicted for the terrace of the Smith House is twenty, seventeen and seventeen for the architectonic, spatial and diagrammatic level respectively and are shown in Figure 5-32.

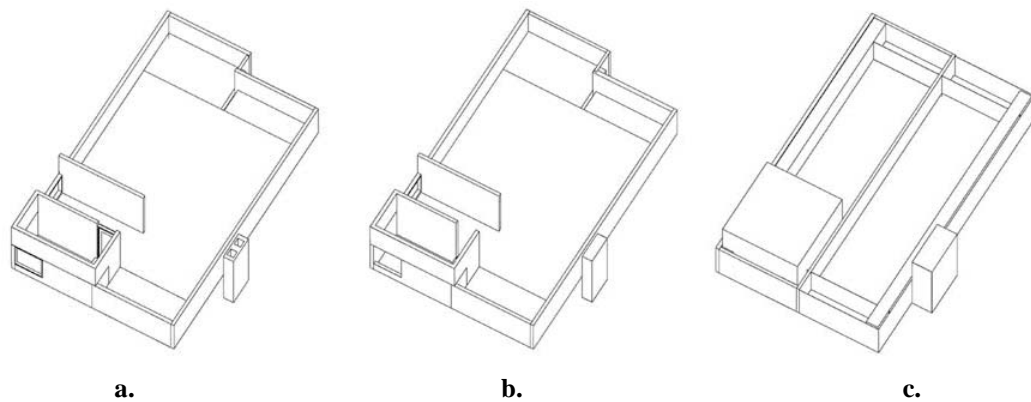


Figure 5-32: Smith House Terrace - notations: a) Architectonic; b) Spatial; c) Diagrammatic.

5.5. Play: Partitions

‘Alongside the visual symmetry, we should imagine another one, out of which the former only ‘shines forth.’ Plotinus in (Panofsky 1968)

The basic mechanism to abstract the elements of the house and foreground their relationships as they are translated from level to level has been put in place. What is interesting in this process is the re-working of the compositional machinery of the design and the exploration of the possibilities that this system allows with respect to the symmetry parts. This section starts with the extraction of the five subgroups of the rectangle in order to identify the symmetric transformations needed for this description. Additional marks concerning shape, weight, and projection are added to yield the organizational scaffolding of the house.

These diagrams provide the underlying structures for the partition of the Smith House into five classes, each corresponding to a unique subgroup of the structure of the rectangle. Each of these five partitions of the geometry can occur at any floor of the house, -first, second or third, and for any level of representation, architectonic, spatial and diagrammatic. The total number of partitioning n then for this case study is forty five distinct ones. Still, what is attempted here uses these methodological tools but in a new context. The key idea here is that the decomposition of the geometry of the house occurs only at the diagrammatic level and this partitioning will specify how the higher level notations would correspond to that. The total number of drawings is still the same as before, forty-five, but there is a major qualitative shift in the analysis of information. A little application of a symmetry analysis on the architectonics notation will produce descriptions primarily characterized by asymmetrical information. The transformed application of the symmetry analysis here of the architectonic level will pick up elements and conditions that would have gone unnoticed before. The complete visual computation of all forty-five symmetry partitioning for the Smith house for each floor is given in the Figure 5-33 through Figure 5-37. All notations are read in sets of nine and are read from the lower right side to the upper left. Each row is read from right to left and each column from bottom to up. The lower row represents the analysis of the first floor, the middle row that of the second floor and the top row the analysis of the third floor. The right column shows the diagrammatic notation, the middle column the spatial and the left column the architectonic. All figures show in the first three entries (a)-(c) the isolated symmetry elements of each symmetry subgroup and in the second three entries (d)-(f) the same symmetries combined with all the higher order symmetries in the lattice for all three floors. This same dual presentation is reproduced through Figure 5-37.

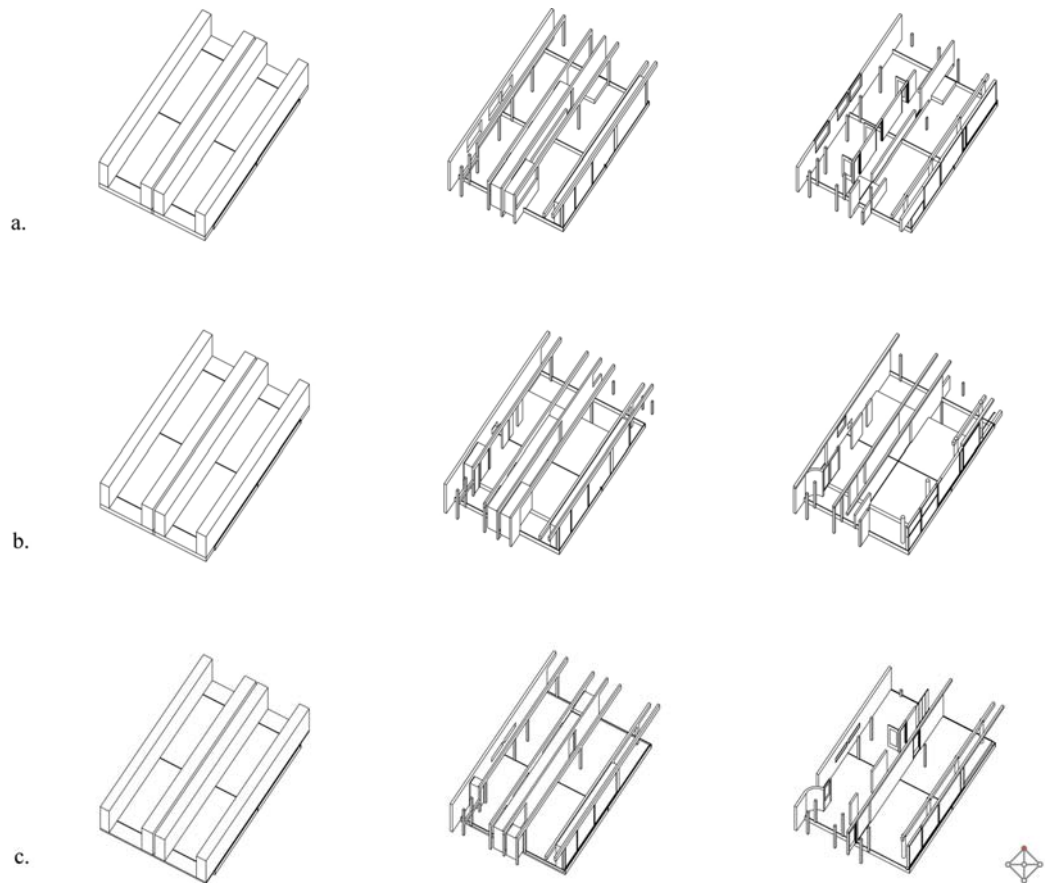


Figure 5-33: Dihedral symmetries of the Smith House at the diagrammatic level and their correspondences at the spatial and architectonic levels. a) Third floor; b) Second floor; c) First floor

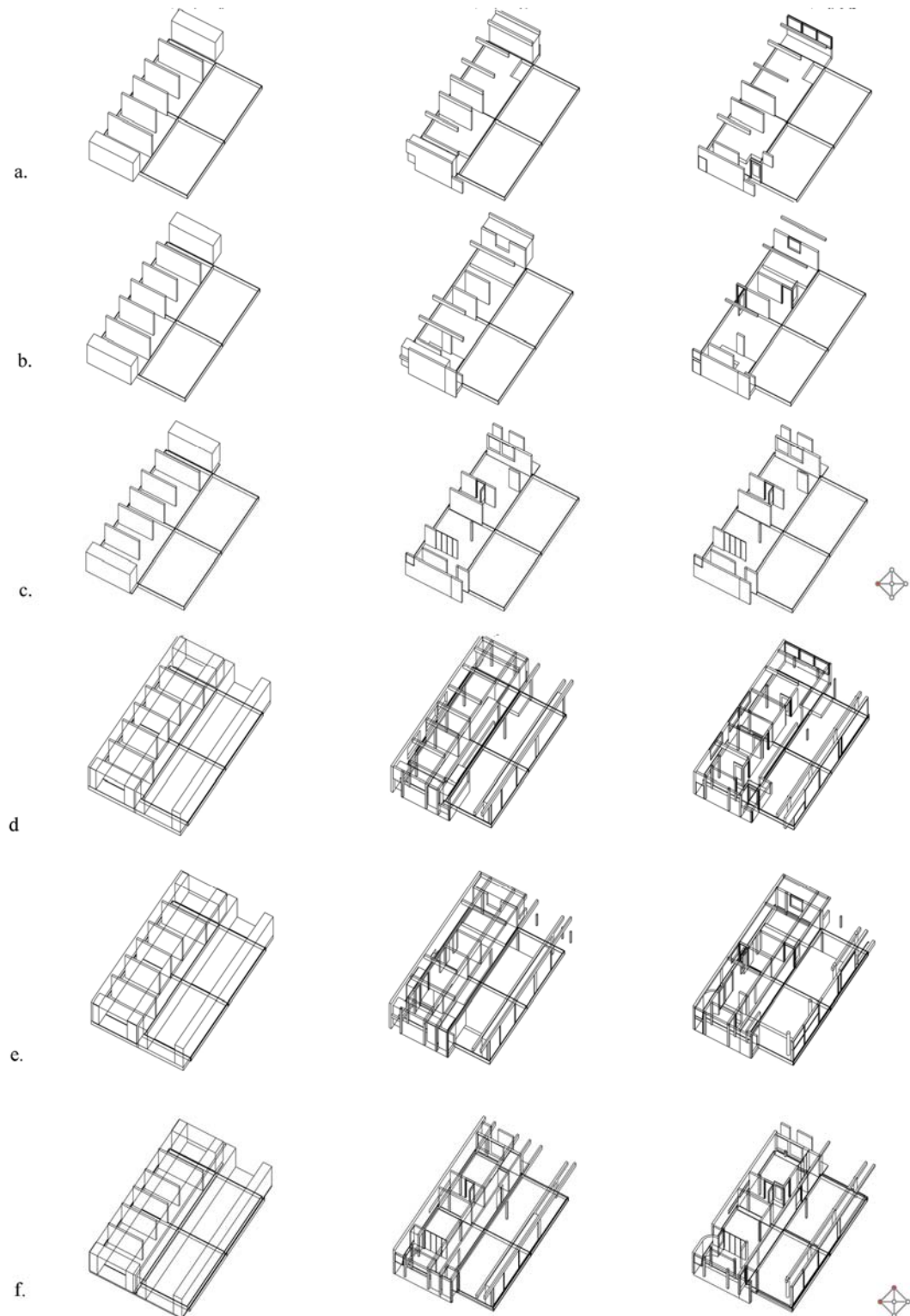


Figure 5-34: Vertical (V) symmetries of the Smith House. a-c) Isolated vertical symmetries for each floor and notation; d-f) Combined vertical symmetries with dihedral for each floor and notation

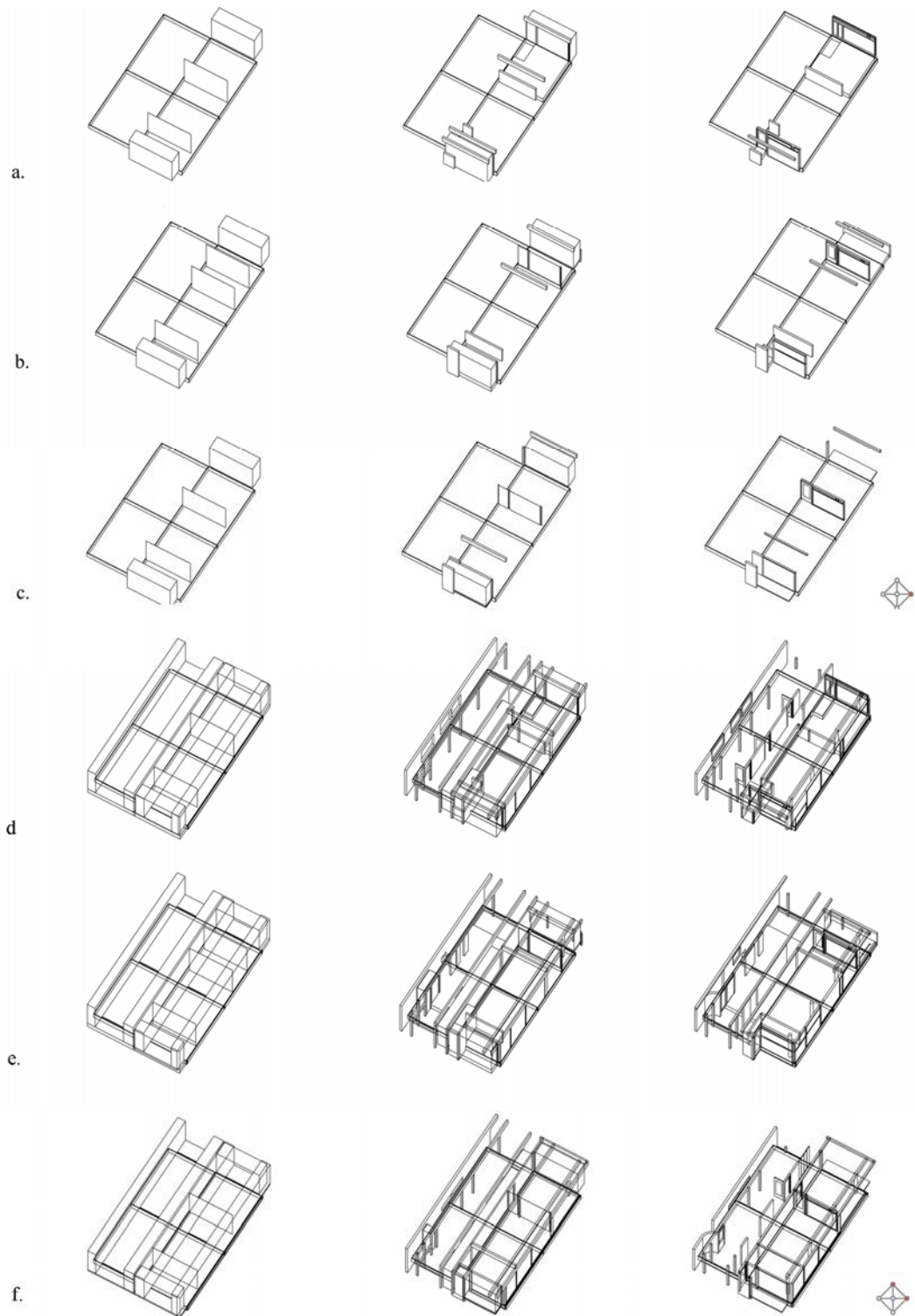


Figure 5-35: Horizontal (H) symmetries of the Smith House. a-c) Isolated horizontal symmetries for each floor and notation; d-f) Combined horizontal symmetries with dihedral for each floor and notation

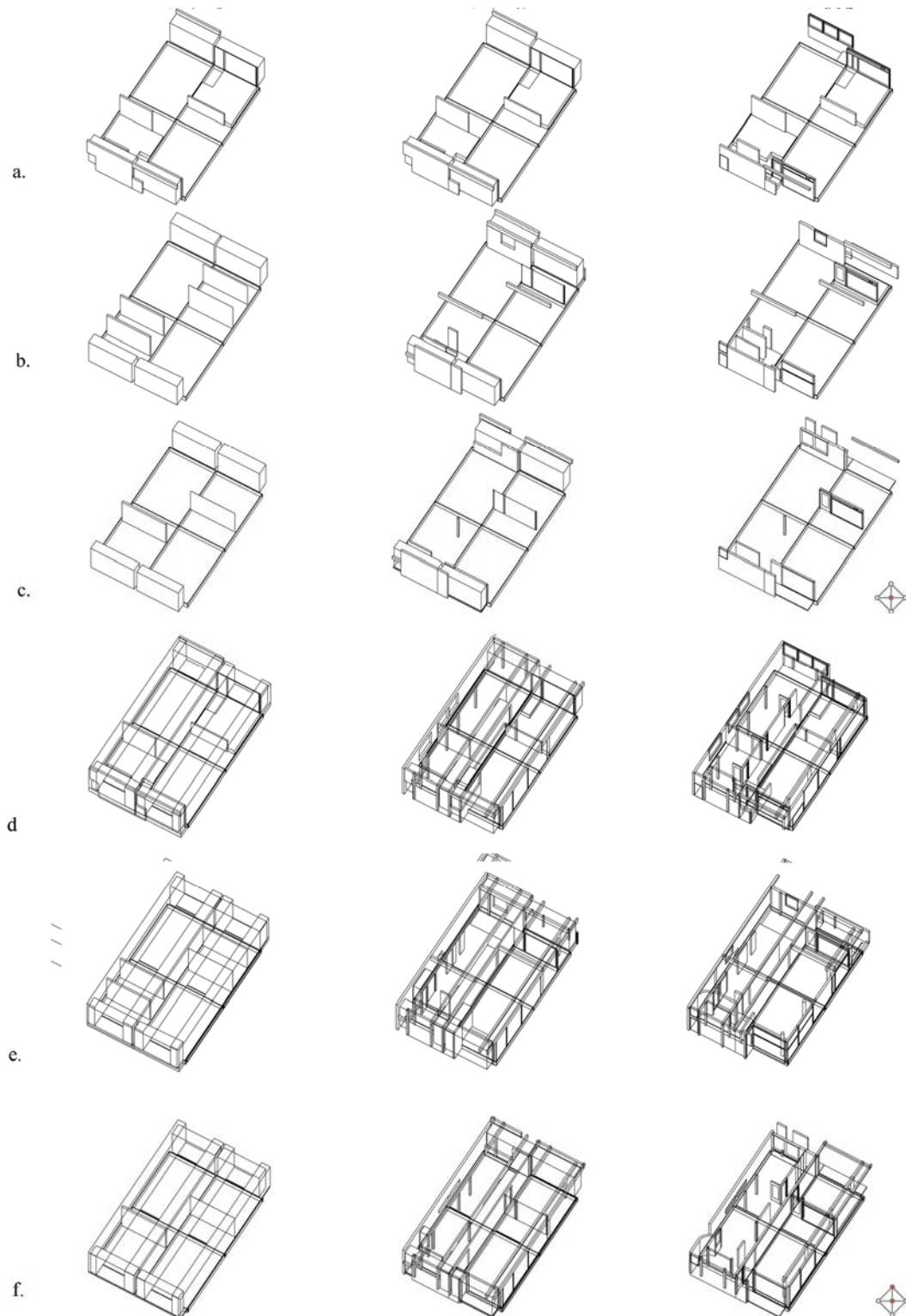


Figure 5-36: Rotational (S) symmetries of the Smith House. a-c) Isolated rotational symmetries for each floor and notation; d-f) Combined rotational symmetries with dihedral for each floor and notation

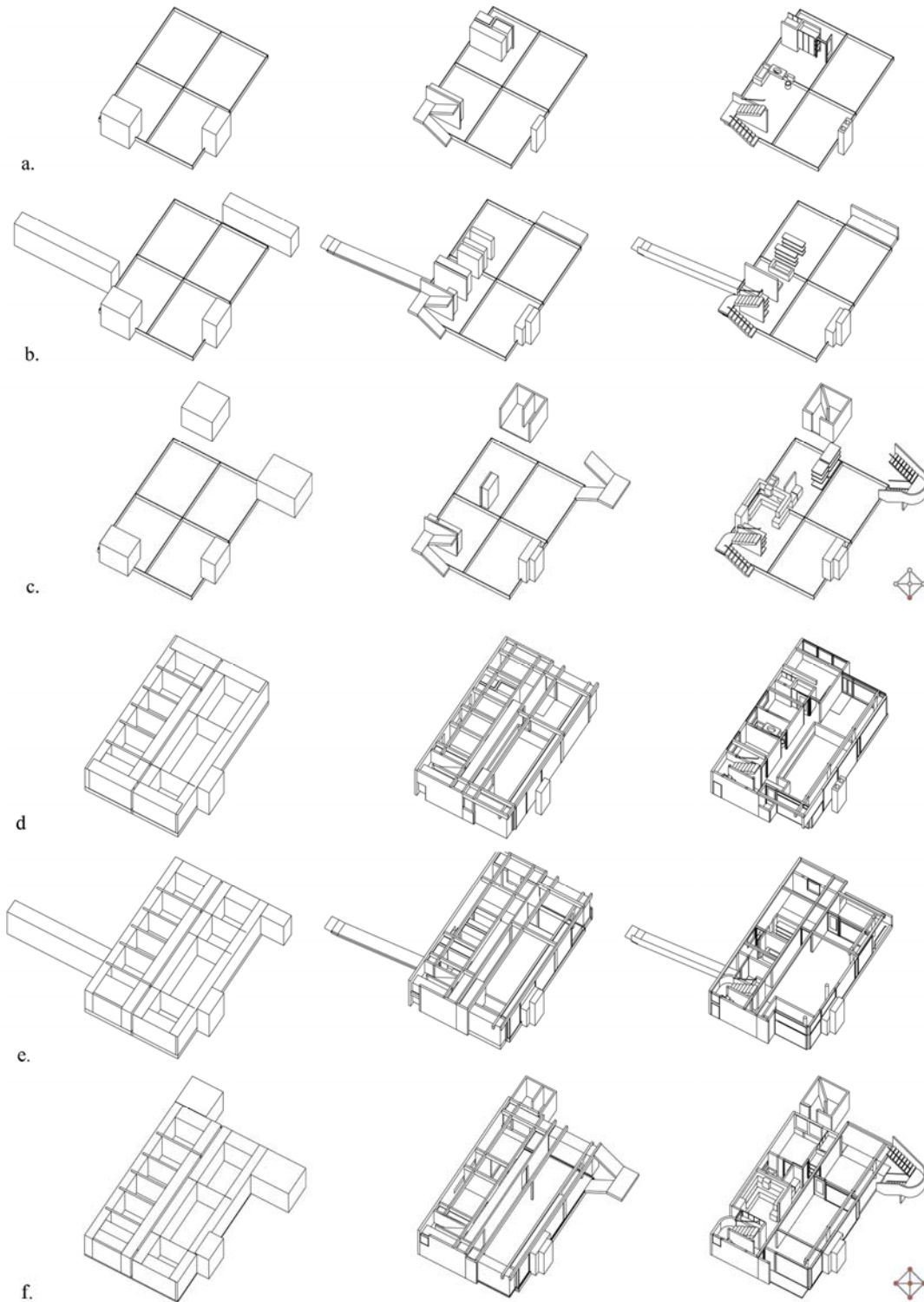


Figure 5-37: Identity (C) symmetries of the Smith House. a-c) Isolated identity symmetries for each floor and notation; d-f) Combined identity symmetries with dihedral, vertical, horizontal and rotational symmetries for each floor and notation

5.6. Fast Forward: Ordering

By extracting sub-shapes which maximize the representation of a particular sub-symmetry or a combination of some of them, it is now possible to construct a partial order lattice or semi-lattice to illustrate the overlay of symmetries involved. With these nested underlying structures for the description of the design, a perceptual interface becomes accessible through non-visual functional relations underlying the visual features appear. The following diagram illustrates all possible subgroups of symmetry: three of them with two elements, and one, the identity, with one element. The structure of the diagram can be accounted for in two ways: from top to bottom, sub-symmetries are subtracted from the full symmetry of the rectangle; and conversely, from the bottom to top, sub-symmetries are added to achieve higher orders of symmetry. Such a reading is analogous to a lattice diagram of a partially ordered set, or sub-shapes of a shape.

The partial order lattice may offer the essential representation of the structure of the Smith House. The representation of this structure by means of modeling requires the determination of levels of abstraction where the operations take place according to the logic of the design. If the lattice reveals the deep structure of the object, one still need to refine the whole semantics that is carried out throughout the representation. Thus, the definition of the levels of abstraction requires that the symmetry operations carry a kind of representation that embeds some semantics. Thus, with three different levels of abstraction, one can generate three levels of detailed lattices. As example, the Figure 5-38 through Figure 5-40 shows the implementation of the typical lattice floor by floor. The spatial notation is omitted for clarity of representation.

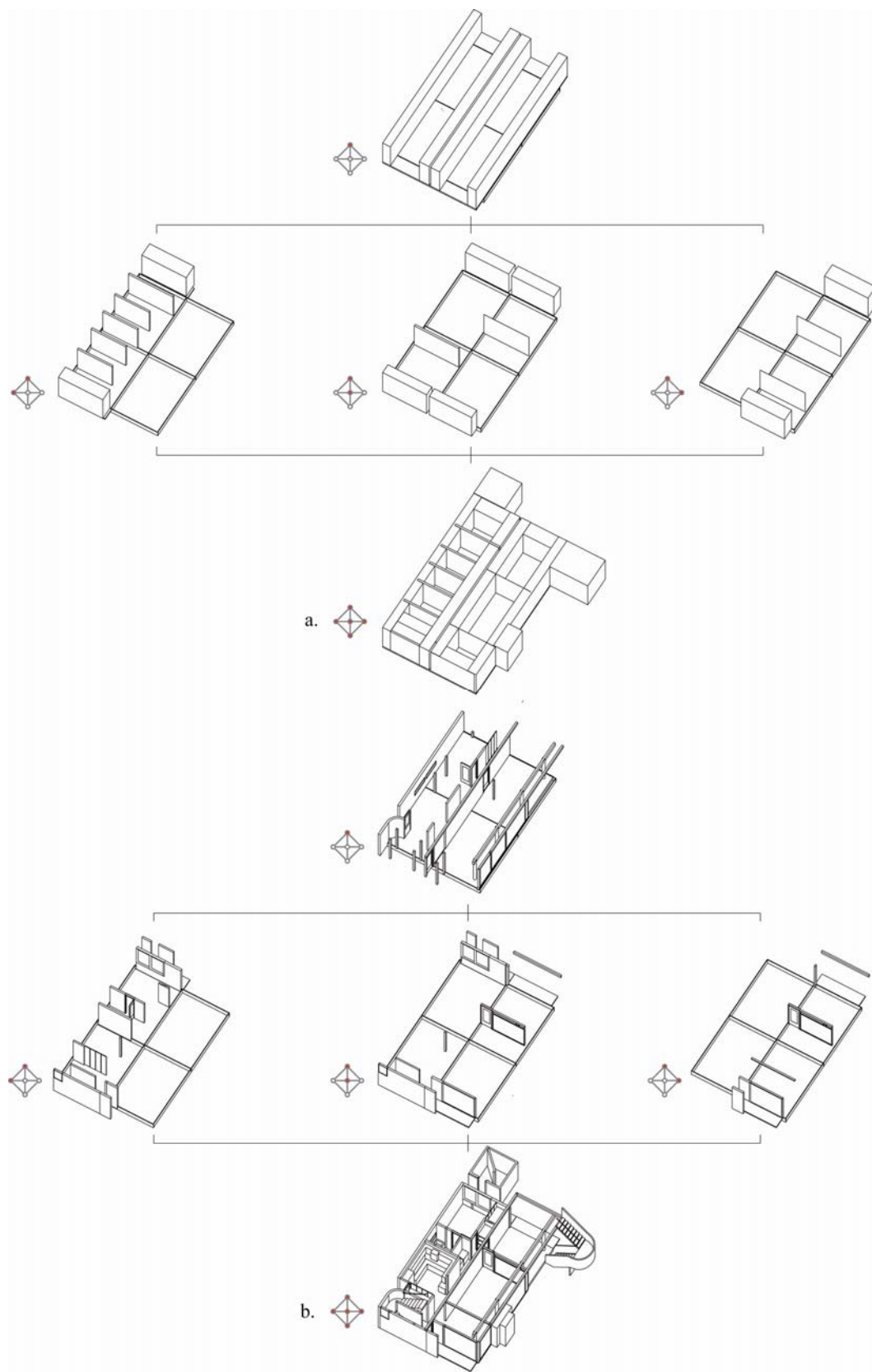


Figure 5-38: Partial order lattice of the first floor: a) diagrammatic; b) architectonic.

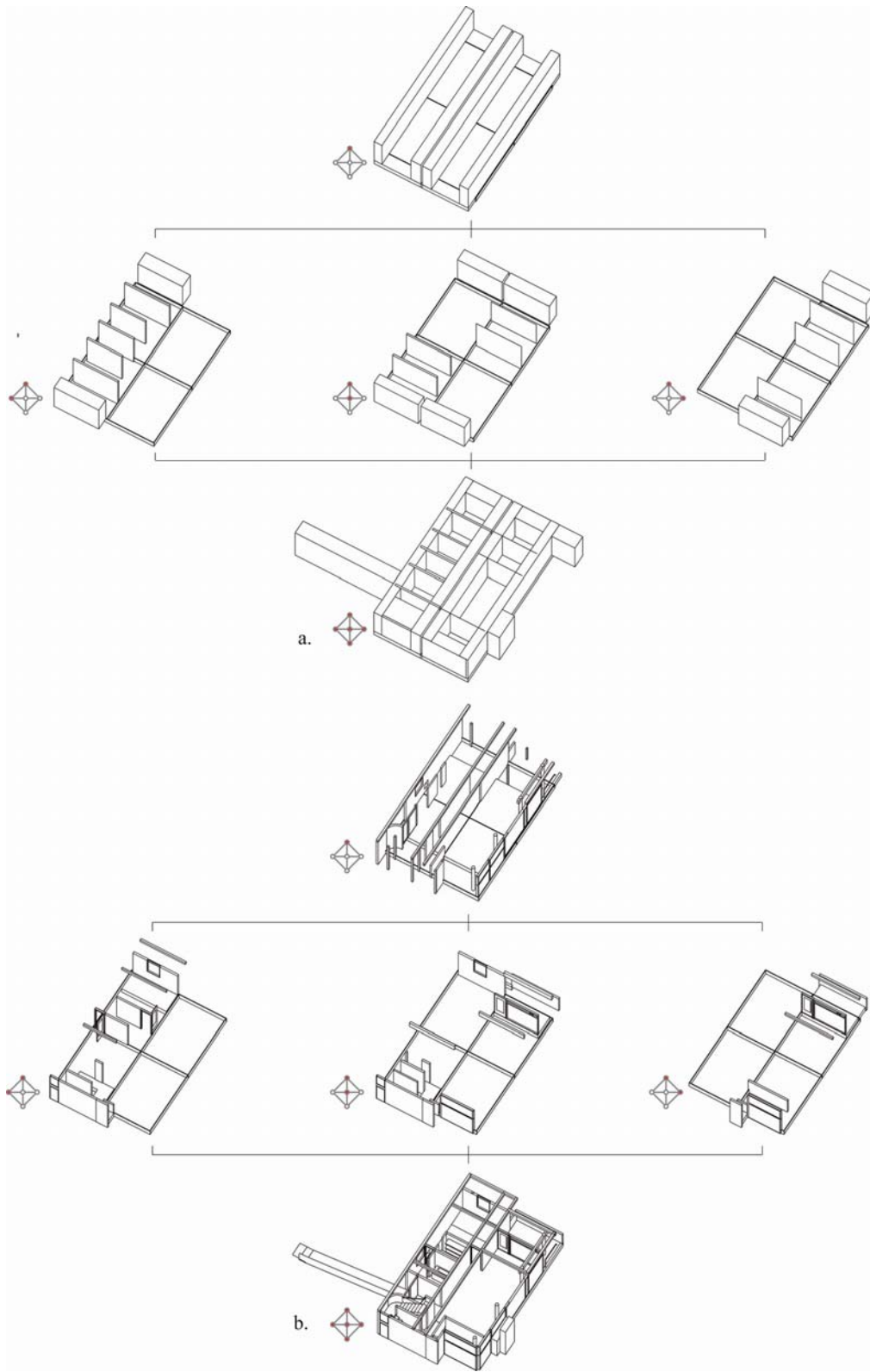


Figure 5-39: Partial order lattice of the second floor: a) diagrammatic; b) architectonic.

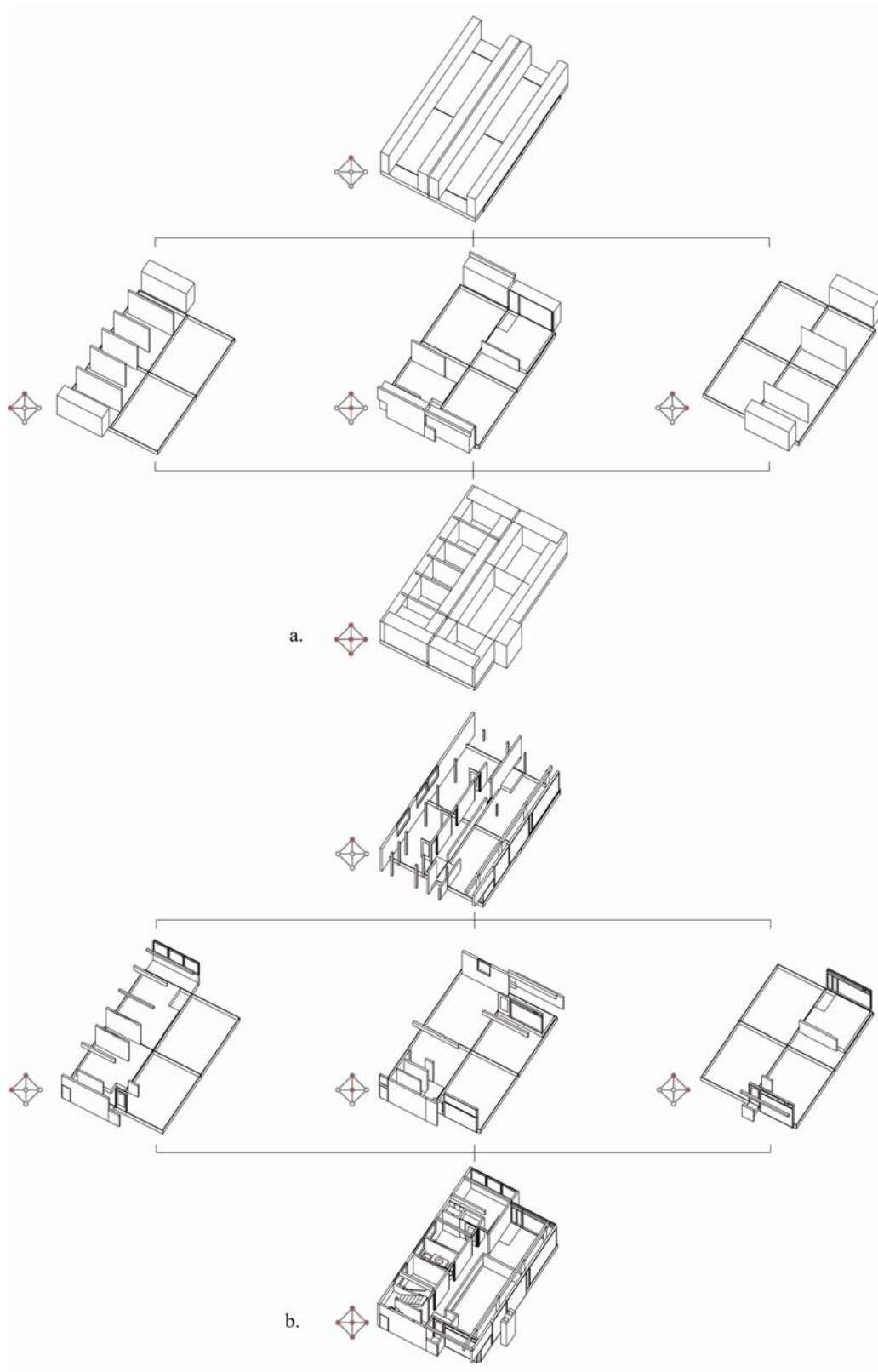


Figure 5-40: Partial order lattice of the third floor: a) diagrammatic; b) architectonic.

5.7. The Smith House recombinant

The sub-symmetry analysis has shown all the possible symmetrical correspondences that can be drawn in the Smith house. Furthermore these relationships were ordered in partial ordered lattices to show how these relationships are nested in specific ways one within the other. Here a somewhat different approach is taken and its major focus is the juxtaposition of all these correspondences, one with the other, to examine partial group theoretic descriptions of the Smith house and in the way of doing so foreground specific relationships that a straightforward application of group theory wouldn't do.

The lattice of the rectangle consists of five subgroups. These subgroups can be combined one with another to comprise a set of $2^5 =$ thirty-two possible design worlds that are differentiated one another with respect to the number of elements that belong in each subset. The number of combinations of symmetry elements r among a set of s elements is given in (1).

$$\frac{s!}{r!(s-r)!} \quad (1)$$

For example for a set s comprised of five elements the number of subsets r comprised of three elements is given in (2).

$$\frac{5!}{3!(5-3)!} = 10 \quad (2)$$

The complete list for all subsets comprised by five elements *including* dihedral symmetries (D), vertical reflections (V), horizontal reflections (H), half-turn rotations (S) and identity transformations (C) is given in Table 6.

Table 6: Complete listing of recombination of subsets of the dihedral group D_2

<i>n</i> -ary element set	Recombination List	#
null set	\emptyset	1
unary set	D, V, S, H, C	5
binary set	$DV, DS, DH, DC, VS, VH, VC, SH, SC, HC$	10
ternary set	$DVS, DVH, DVC, DSH, DSC, DHC, VSH, VSC, VHC, SHC$	10
quaternary set	$DVSH, DVSC, DVHC, DSHC, VSHC$	5
quinary set	$DVSHC$	1

Furthermore all thirty-two sets – design worlds can be augmented each three representations corresponding to the three floors of the house, making then a total of ninety-six drawings including the empty set and these drawings can be given in three-different versions for the architectonic, spatial and diagrammatic notation bringing the total number of representations to two hundred and eighty-eight. This recombinant vision exhausts all possible ways that the parts of the house can be combined and should therefore be able to capture all theoretical statements on symmetry that have been said or could be said about the house. The thirty two sets of subsymmetries of the dihedral group is given here in a pictorial way as well to emphasize the notion of order of all sums subsymmetries within the partial order lattice in Figure 5-41.

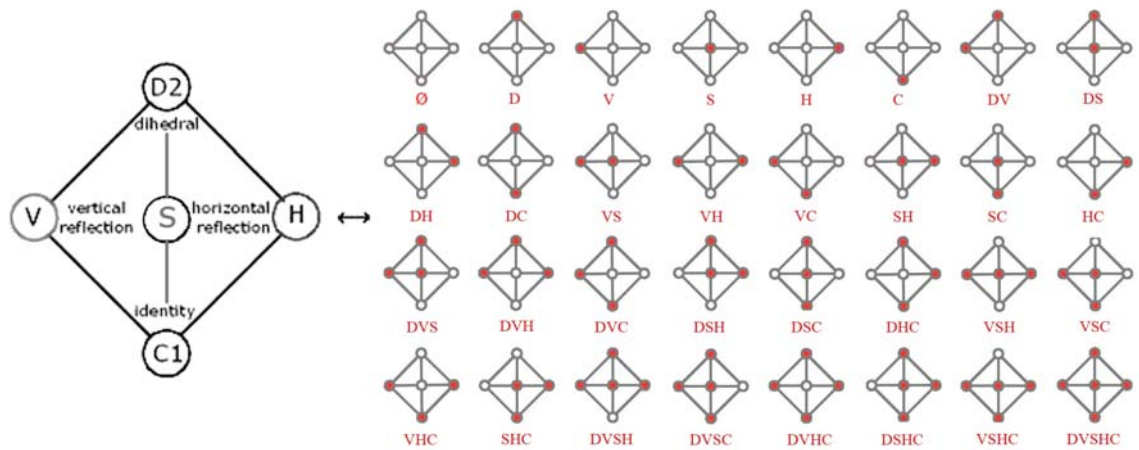


Figure 5-41: A complete list of the diagrams of all combinations of symmetry parts

One of the most important principles of modernism – and neo-modernism is its overt negation of the obvious; symmetries are to be avoided if the only thing they do is to bring attention to themselves. In architectural discourse, this refers to the idea of de-familiarization (Shklovsky 1917). Still, it is argued here that this partial, incomplete and ambiguous rework of classical principles of compositions such as symmetry, can indeed be discussed in terms of the very same tools that describe the straightforward applications of these tools. The goal for analysis is to extract the parts of the design that foreground qualities that are hidden within the structure of the design. This leads to a selection of a ‘minimum building element’ to start with. Essentially it is possible to select the clearest possible image of our design intent as a whole. This minimum part may be composed of any single subgroup or combinations of subgroups of interest.

A sample of this approach is given where the subset of the house consists of three groups of isometries, the complete list of dihedral symmetries (D), the rotational symmetries (S) and the identity symmetries (C); this part is described here as DSC and is presented to foreground the individual values of parallel layering (D), rotary movement (S) and collage elements (C), but at the same time juxtapose the qualities one against the other and examine how the presence of the one clarifies or obscures the significance and role of the other. The visual model-prototype is interpreted using two distinct descriptive systems: facts and values, or forms and functions. Here forms are represented by recombinant DSC. Functions are suggested by values such as: parallel layering (D), rotary movement (S) and collage (C). A relation λ represents the mapping of one system to another with logical variables 0 and 1. Using this relation, like Ho (1982 d), a similarity relation permits overlap between parts, whereas the equivalence relation separates parts into disjoint classes. The lattice of DSC represents a partial order relation. It preserves within subgroups similarity and between subgroups mutuality. A weighted relation may suggest that at lower level, {D, S, C}, at intermediate level, {DS, DC, SC}, and at top level {DSC} where \emptyset is included.

This interpretation of a randomly chosen recombinant gives some hint on how these recombinants can be used toward synthesis as a visual prototype to start with. It is also important to remember that the isolation and foregrounding of specific subsets of symmetries happens in the diagrammatic level and all lessons learned transfer then to the architectonic level. The DSC mode for all three representations discussed so far, namely, the architectonic, spatial and diagrammatic notation, is given in Figure 5-42.

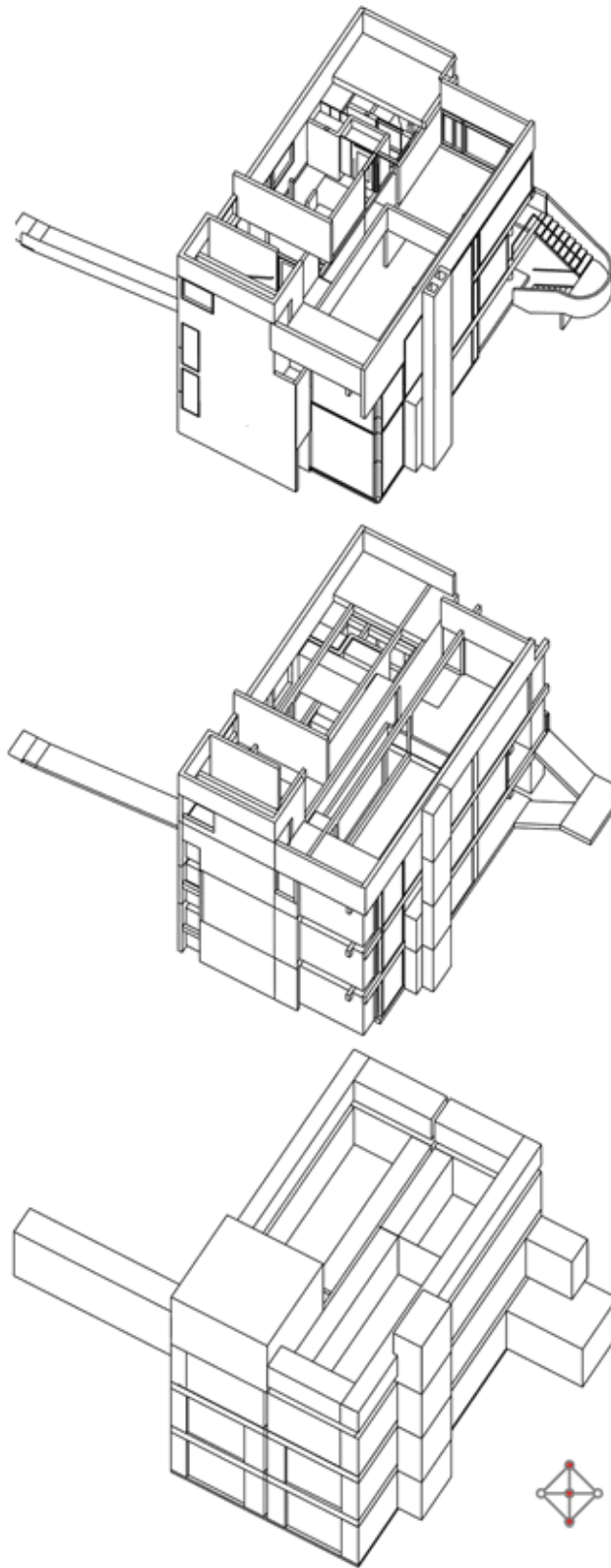


Figure 5-42: One of the ninety-six case studies: DSC

A nice outcome of this constructive combinatorial approach is its ability to capture and reflect on existing debates on formal analysis of the house. For example Frampton in his paper “Frontality vs. Rotation” suggested that most works of the NY5 architects are characterized by the common theme of simultaneous frontal and rotational development of composition but furthermore suggested that Meier does not manage to resolve the intrinsic conflict of the two systems. Rosemarie Bletter (1976) critiques Frampton that his categories of frontality and rotation are in the end too broadly defined and too general to help precise analysis. The DSC model, and in fact several other subsets of the building can help investigate questions and criticisms like those Bletter suggests. The applications of the two categories in the analysis of buildings are somewhat non-systematic; she further claims that in Frampton’s paper frontality in Heiduk’s projects is used to refer to overall massing, in Graves’ work is used to refer to the entrance while in Eisenman’s and Meier’s work is used to describe the interior gridding of the space. These properties are clearly foregrounded in the representation suggested here. It is clear for example that the partition of the house in terms of frontal symmetries – in the original model the H symmetries – and the corresponding partition to the side symmetries, in the model the V symmetries, show clearly the part of the design that is subject to these transformations and mostly the embedded relationship of the rotations to the two systems of reflection. Furthermore the relatively dense compartmentation of the design given the dihedral, rotational and identity transformations, suggests that the house is conditioned in a great extent by both orthogonal axes and not just one. In conclusion, it is suggested here that the partial and incomplete correspondences that may be observed in the Smith House should be captured in any of the ninety-six subsets of the complete set of transformations of the smith house. The complete list of all drawings for all floors and any recombination is given in Figure 5-41 through Figure 5-49.

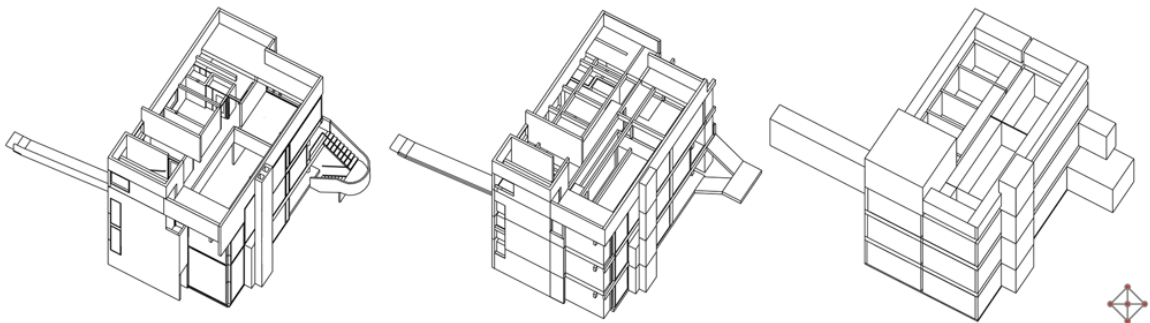


Figure 5-43: A unique combination of 5-element symmetry parts: DVSHC

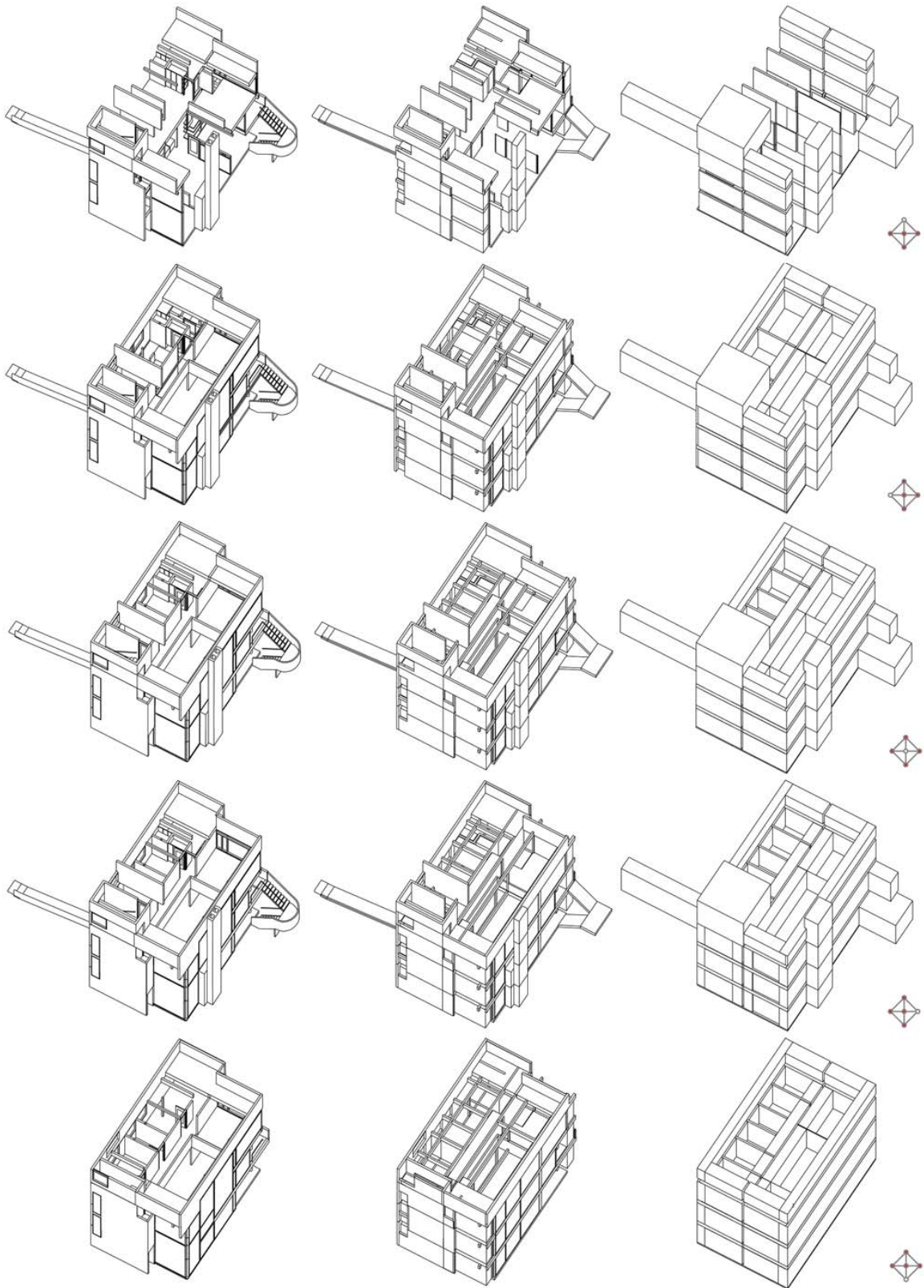


Figure 5-44: A list of all combinations of 4-element parts: VSHC | DSHC | DVHC | DVSC | DVSH

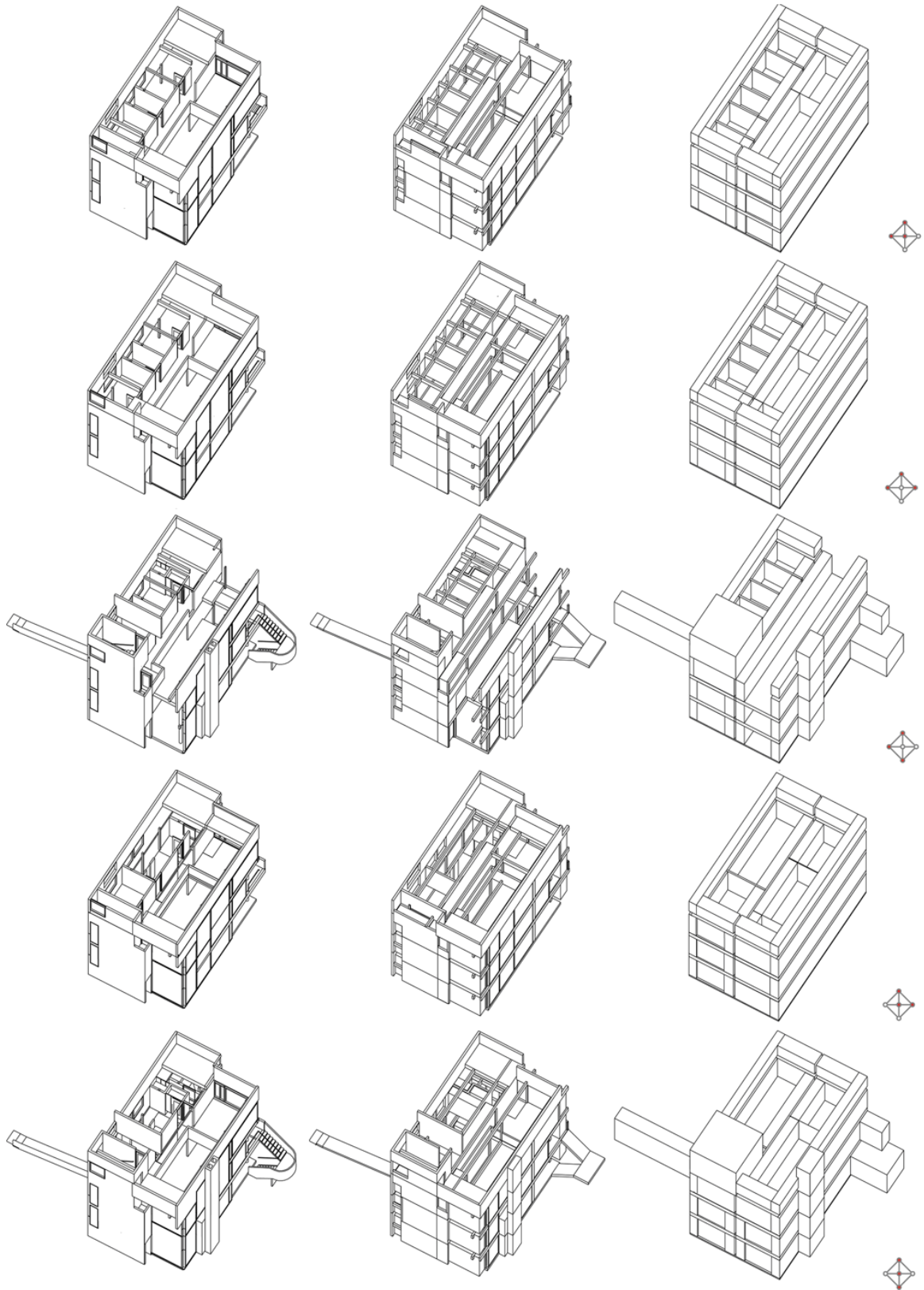


Figure 5-45: Combinations of 3-element parts: DVS | DVH | DVC | DSH | DSC

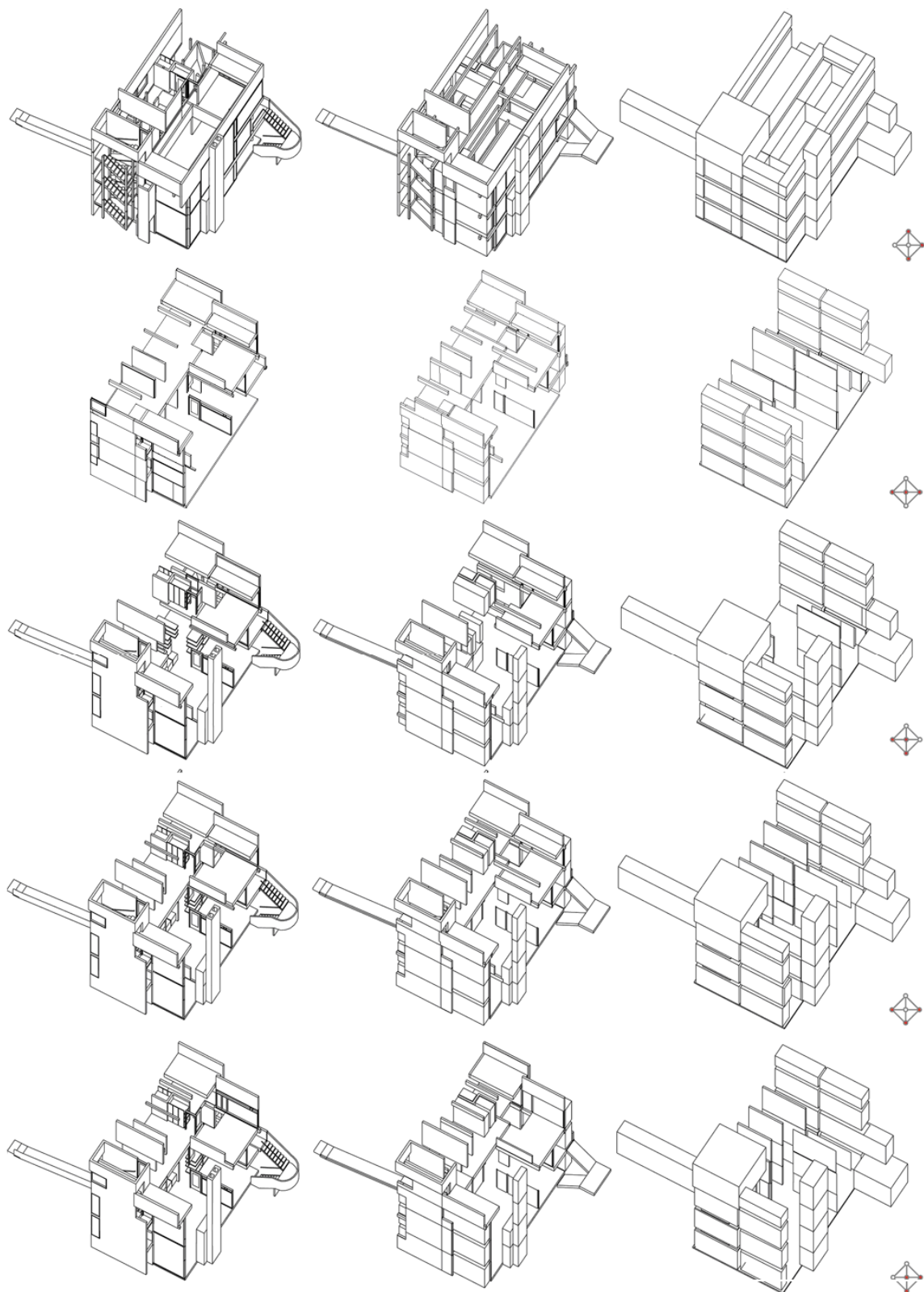
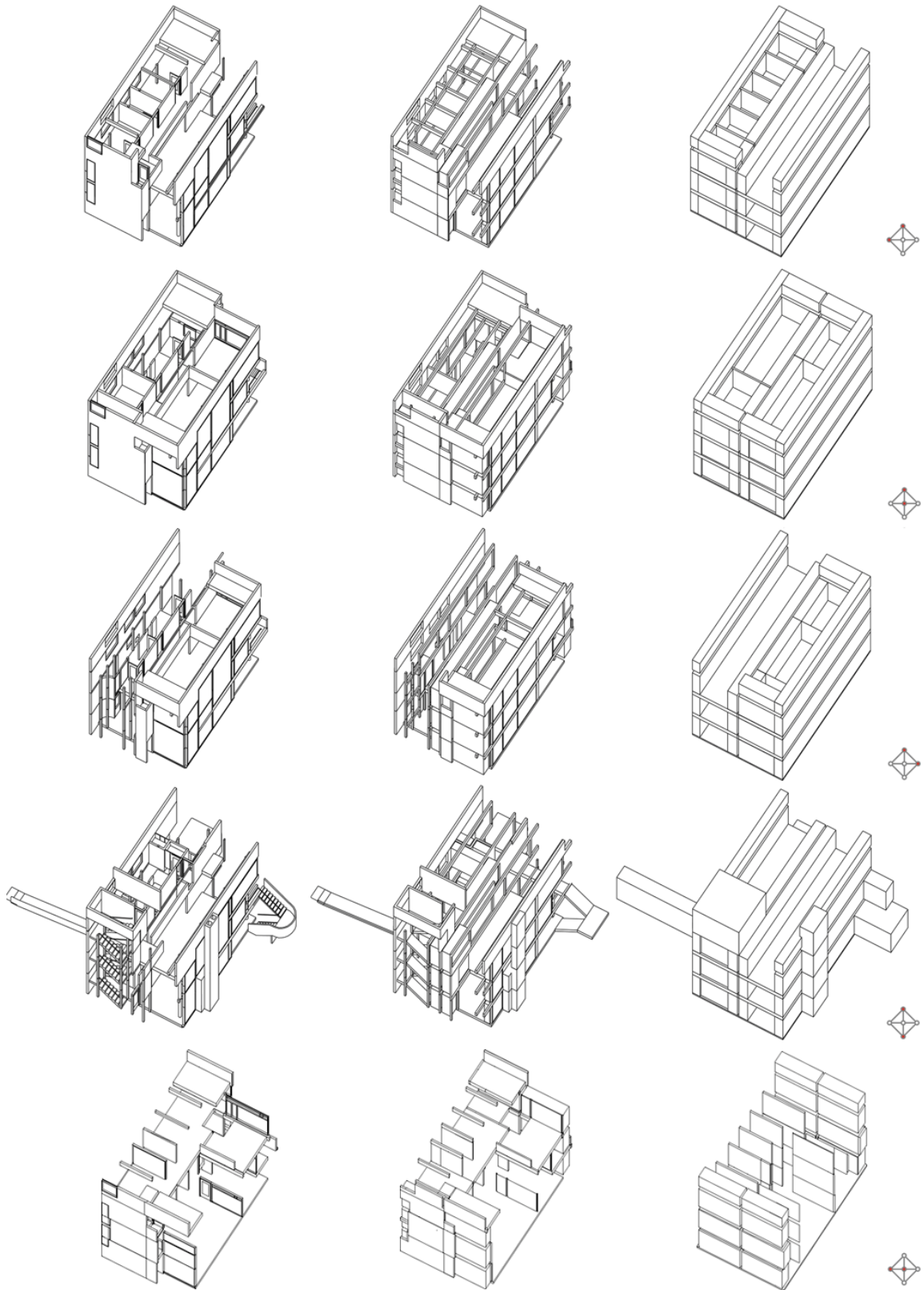


Figure 5-46: Combinations of 3-element parts: DHC | VSH | VSC | VHC | SHC



F Figure 5-47: Combinations of 2-element parts: DV | DS | DH | DC | VS

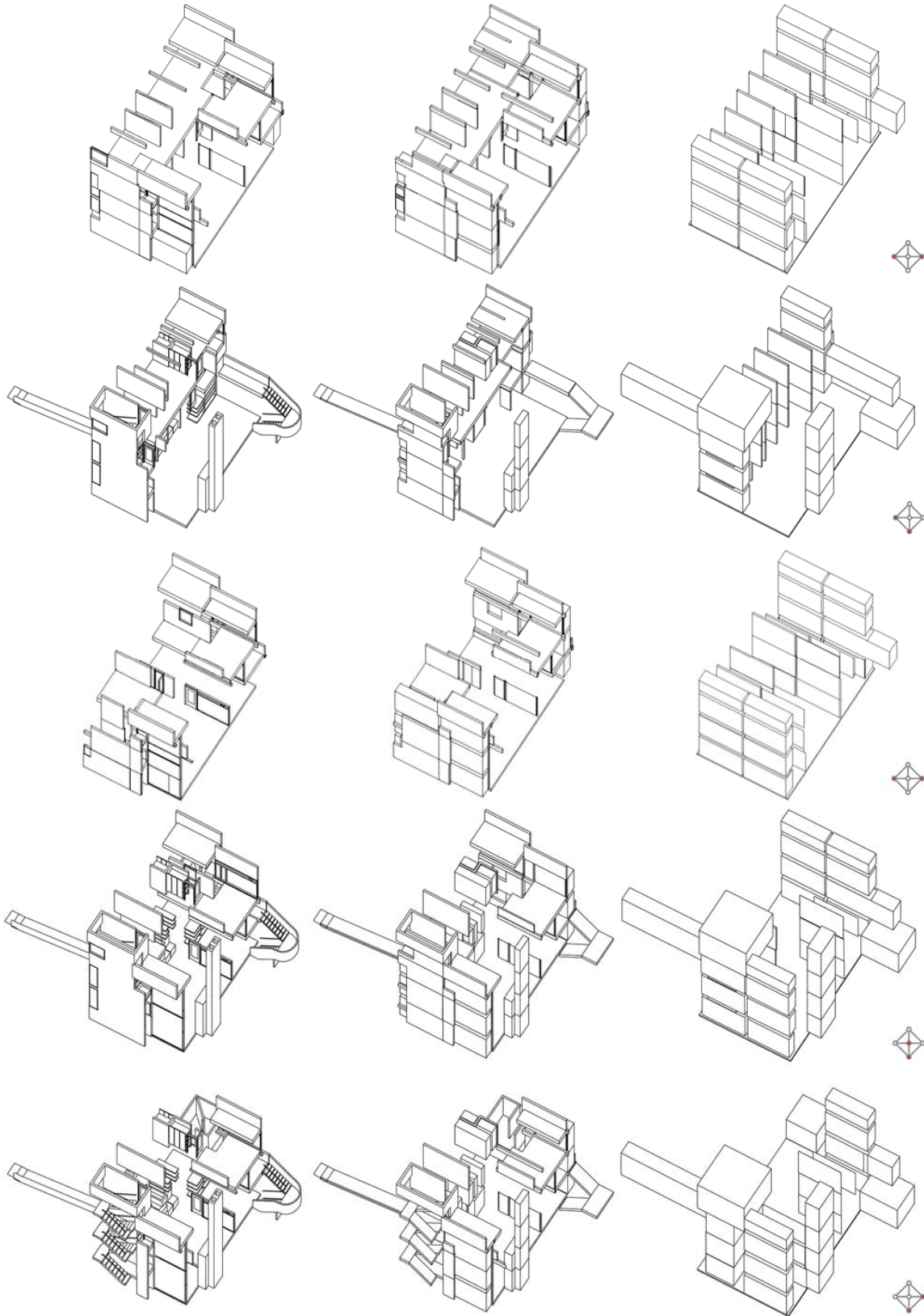


Figure 5-48: Combinations of 2-element parts: VH | VC | SH | SC | HC

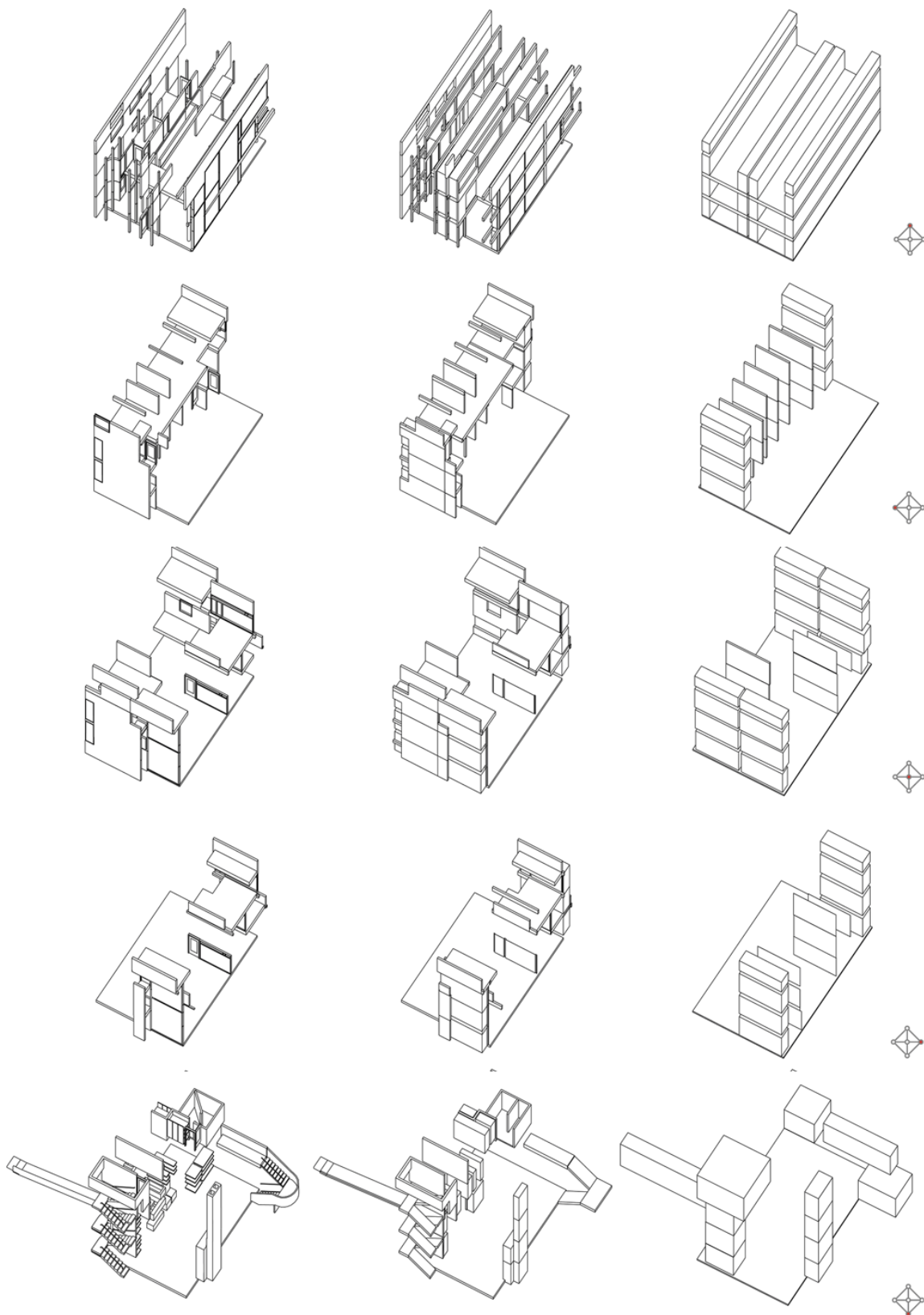


Figure 5-49: A complete list of 1-element symmetry parts: D | V | S | H | C

5.8. Discussion

An application of the methodology of formal analysis was attempted here using Richard Meier's Smith House as a case study. The history and some competing analytical approaches were presented in the first part of the chapter and the in second part the preliminary findings of the analysis were e to produce a formal model of the house. Three levels of notations were used to tackle constructs of composition such as layering, transparency, and collage. All plans of the house were represented and decomposed in specific ways as described in the previous chapter and the computation of all symmetry parts took place in entirely visual terms. A final reassembly of the layered symmetries explained the structure of the symmetry of the house and provides an illustration of one of the basic arguments of this thesis on the foundation of a theory of emergence based on symmetry considerations.

A major challenge for the evaluation of the analysis is the degree that the decompositions provided proved visually the established discourses on the house. The degree to which these representations corroborate existing discourses on the Smith House provides a more contested territory and this is an aspect of further critical research on the ability of the proposed methodology to align itself with existing analytical discourses and prove them or not. Here the implementation on Meier's Smith House has nicely demonstrated a simple but quite significant fact, the possibility of using formal tools from group theory and lattice theory to discuss symmetry properties of designs that do not yield immediately their underlying structure.

Chapter 6 Epilogue and future research

Chapter 6 provides a summary of the work, an assessment of its strengths and limitations and suggests future work.

6.1. Introduction

'Many things, both new and old, my dear Cube brings into view; so my Cube much pleases me, because through it so much I see. It is a little world.' Froebel cited by (Downs 1978)

A major motivation for this work is the systematic exploration of the simplest of means present in a design inquiry. Many other tangential discourses were informed by this precise look and grew in several directions to acquire a life of their own. An example is the suggestion that languages of architecture are often informed and constructed by, with the same formalisms that can be used to describe, interpret and evaluate them. For example, Richard Meier's work has been presented here as a hyper-refinement of the modernist imagery that has been inspired not by machines but by other architecture that was inspired by machines and especially Le Corbusier; similarly, the group formalism that can describe Meier's architecture could constitute a hyper-refined construction that relies on specific representations and mappings that foreground internal complex relationships of the structure itself, i.e. the symmetry subgroups and super-groups of any given spatial configuration. This analogy far as it goes has its limitations too, and the same exist for many other implicit theses herein. Here in this last section is an attempt to foreground a series of other extensions and domains that this research points to. These extensions generally fall into two categories; a) on the improvement of the system itself; and b) on the interpretative capabilities it affords for the construction and evaluation of critical languages of design.

6.2. Model

The model described in this research uses well-known constructs from group theory along with specific constructs of architectural notation including abstraction, projection, weighting, and layering to construct representations that are then computed in the system. Four major avenues for future research are readily available and all of them have to do with the types of representations fed into the system and the transformations under which the rules apply in the system. All these directions are briefly described below.

6.2.1. Dimensionality

Currently, all plans of a design are analyzed in terms of corresponding group structures and all are represented in partial order lattices using axonometric orthographic projections. A major new direction for the model would be to tackle three-dimensionality both in the representations used as in the group structures that provide the partial order lattices. Two alternative schemes are readily envisioned; one would still use the planar representations of the current model but would require a parallel computation of at least three planes to simulate the *XYZ* Cartesian space. In this case, all drawings computed are two-dimensional as are the groups that describe them, but their combinations produce the three-dimensional analysis. An alternative mode would use directly three-dimensional representations and three dimensional groups. In this case, the corresponding group that would describe the symmetry, say, of the Smith House, would be the $C_2C_2C_2$, a three-dimensional prismatic group consisting of the Klein group C_2C_2 of order 4 augmented by a cyclic group C_2 of order 2 (Economou 2001). The degree that an analysis can be carried directly in three-dimensional space is really an open question as the complexity of the structure of these groups is growing exponentially with the rising order of the groups. Still, the argument is valid because most of the group structures dealing with the design are relatively of low order. A table of three-dimensional structures that capture the symmetry of three dimensional shapes that have a singular axis of rotation n and for order $n < 12$ is given in Figure 6-1 (Economou and Grasl 2007).

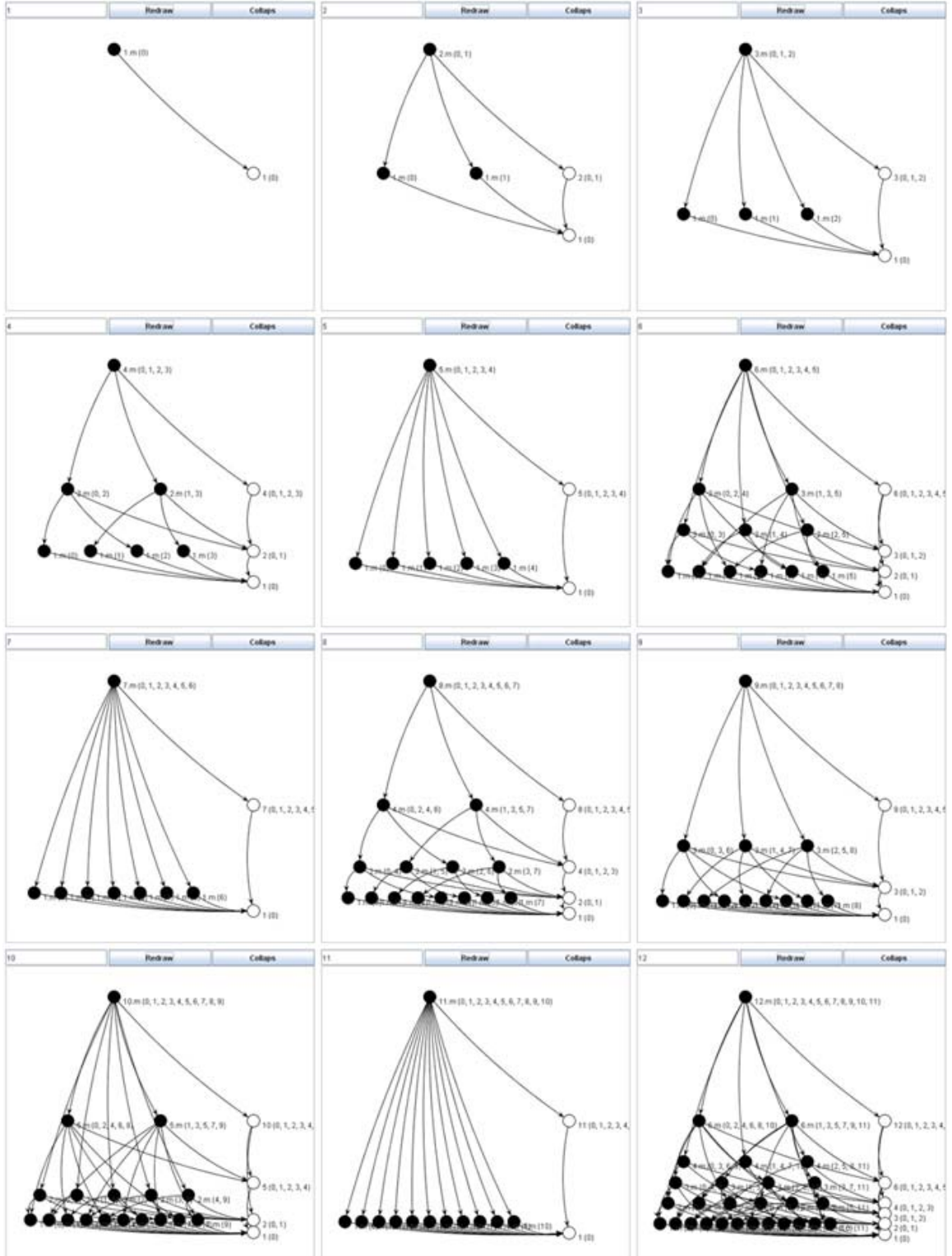


Figure 6-1: Graph representation of the subgroup structures of the dihedral groups D_n for $n < 12$

6.2.2. Topology

The existing model is used primarily on rectangular compositions that are conditioned by straight lines and right angles. The enlargement of the model to include designs defined in affine, linear or topological worlds is straightforward. The rules that map an initial complex design to a simpler rectangular one can be straightforward for a good amount of cases. In these design worlds the nested hierarchies can be modeled after corresponding hierarchies of transformations. For example in a Euclidean system – the current one of the model – any rectangle can be mapped back to a square; this was anyhow the underlying theme for the application of Polya's theorem of enumeration for all wall structures based on a 3×3 grid. More complex design worlds are readily available. For example any parallelogram can be mapped back to a square in an affine design world; any trapezoid or quadrilateral can be mapped to a square in a linear design world and any disk of topological genus-1 (no holes) can be mapped back to a square in a topology or rubber sheet design world. In all these cases, it is straightforward to design mappings between grids by similarity, linearity and topology transformations in conformal mappings and attain more sophisticated regulating lines to support the design systems. A nice illustration showing the same mappings between equivalent systems is given in Figure 6-2 [after Klee in Spiller (1961)]



Figure 6-2: Grid mappings by linear and topological transformations

6.2.3. Shape grammars

The bias of the model described in this work is overtly analytic: given an architecture work the model produces a partition according to a set of symmetry considerations. The exact opposite is a somewhat different problem but even more challenging and rewarding. Given a set of symmetry considerations – perhaps observed in one or more architecture works, the model should produce alternative architectural works that exemplify the given requirements. Sets of shapes and spatial relationships could be carefully selected too to establish degrees of conceptual and aesthetic clarity and vicinity with the original. In these latter cases the reworking of this material can

provide a rich palette to visit not only the composition of the house itself but to contemplate on the possible configurations that are not used in this specific case but can be used in other cases. If one was to use the example worked out in this work one should claim to then to design configurations that could be done either by Meier himself or any other of the NY architects. The range of these applications is indeed unlimited and may even help connect various aspects of group theory and provide new material to shape grammar discourse (Stiny 2006), (Knight 1994). Figure 6-3 shows a partial order lattice exemplifying the group $C2C2C2$, using sets of shapes and spatial relationships broadly conceived as extracted from the NY5 language (Economou 2001).

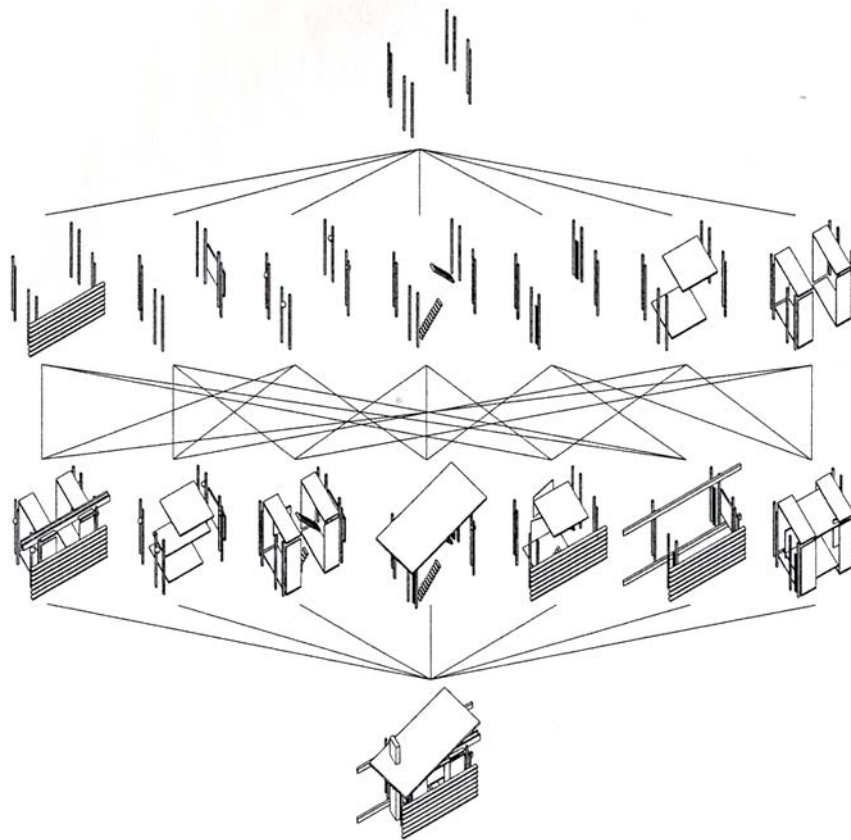


Figure 6-3: A partial order lattice of a design modeled on the $C2C2C2$ structure

6.2.4. Automation

The existing model used computer projections of a three-dimensional model to specify all the parts for the computations. All symmetry parts and projections were manually extracted and arranged in spreadsheets for viewing. An extension of this application for an automatic extraction of these drawings is highly desirable. The extraction of the symmetry part is not a formidably computational problem, albeit not a trivial one. Figure 6- shows a flow chart for the determination of the symmetry group of a two dimensional figure. Corresponding flow charts from three dimensional shapes are readily available too. What is harder is the identification of the parts that need to be examined for symmetry decomposition (Rose 1981).

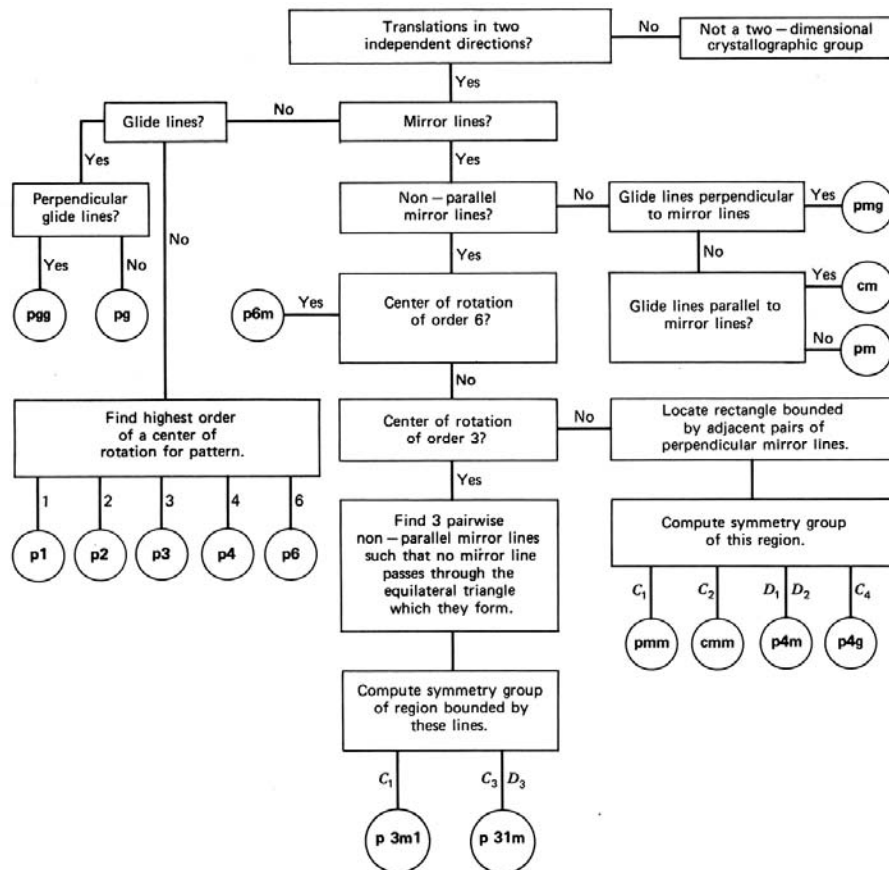


Figure 6-4: An algorithm for identifying the seventeen symmetry groups of the plane

6.3. Interpretation

The model aspires in complete computations that provide all the corresponding parts that are extracted from a given system taken into account the symmetry properties of the underlying configuration. The degree to which the computations support existing discourses about the interpretation of a design work or point to new ones is still a problem to be investigated. Some directions pointing to these directions are given below.

6.3.1. Emergence

The concept of emergence in creative design provides one of the most exciting problems in formal composition and in the formulation of computational models including shape grammars (Knight 2003). 'In addition, the domain of constructive analysis reinforces the term 'conceptual emergence' by strengthening a kind of emergence based upon the exploitation of conceptual knowledge. The fit between visual images stored in the analyst's associate memory and the way these images are mapped into a formal-configurational schema is here defined as conceptual emergence. Images can contribute to the emergence of generic patterns, or schemas. With these assumptions contradicting the idea of unanticipated emergence, it is proposed that domain knowledge guides emergence and that all emergence is, to some extent, guided.

In perception one sees objects that are physically present. In imagery one can 'see' objects that are not currently being viewed. 'Transformational emergence' is the externalization of retrieved images and the activation of transformational operations as a class of design knowledge. Transformation is the ability to modify patterns in images. Objects can be shifted and rotated and alter imaged patterns. Transformations are important in design generation. In order to be able to 'think' with a visual image it is necessary to identify its generic qualities: the 'know-how' to transform.

The domain content of visual images, or visual prototypes, constitutes a significant class of visual knowledge of the designer. The guiding role of these visual prototypes may be said to introduce a

dimension of 'anticipation' in the process of emergence. The designer may not know exactly what he or she is looking for, but it is still possible to select a language for transformations. To this extent, emergence is guided and anticipated. It is the re-cognition of images as visual prototypes which enables emergence. One refers to this kind of guidance function in emergence as 'anticipated emergence' (Suwa, Gero et al. 1999) . Any theory of creative discovery through emergence must be made to accommodate the idea of 'anticipated discovery'.

The visual exploitation of shape ambiguity is an integral part of thinking with images. Designers do not know what exact shape will emerge but they do know how to manipulate shape ambiguity and transform images in order to obtain a desired form since they know in advance the spatial effects of the qualities of subgroups. When designers employ a certain class of images, they canalize emergence. They do not know exactly what form they will see, but they anticipate, and are ready to perceive, a new form. Thus the application of certain symmetrical principles can guide the emergence of the form.

The modeling of emergence demonstrates how cognitive emergence operates and how high-level cognitive schemas contribute to our ability as designers to generate new forms through the manipulation of shapes and images. The specific case study illustrated here is the 'the Smith House'. The following figures illustrate an example of the exploration of the configurative pattern.

6.3.2. *Complexity*

A major idea pursued in this work is that the typical notion of complexity understood as opposite to symmetry, can be cast in entirely different light as an aggregate of simple symmetry constructs or layers. In order to develop such a theory of complexity, a theory of symmetry is necessary to support such claims. One may acknowledge that up to now in the classical tradition, the effects of geometry have been limited to the use of grids, proportions to bind relationships into signifying systems. As suggested by the square-grid composition, what this work proposes is to expand the use of geometry to group theory of transformations in order to map it to any architectural composition no matter how difficult the difficult whole and conflicting geometric systems have to

be. It happens that March and Steadman have renewed the interest toward this approach more than three decades ago with their book, *Geometry of the environment* (1971).

An object has symmetry if there are spatial transformations that allow the object to move, and yet end up occupying the initial space. Group theory of symmetry lets us know how many sub-symmetries of an object can be exploited in design. That is to say a whole design may have no overall symmetry; such a globally asymmetrical composition is always replete with local symmetries or sub-symmetries. Form suggests reference to both internal structure and external outline, and the principle of symmetry brings unity to the whole. Today's design is mainly based on transformations of some visual prototype. The study of Meier's architecture is just one striking example; it teaches the designer that equipped with the powerful tools of computational symmetry, he or she can generate enough configurations to interpret, to generate, and to manipulate form, how complex it may be.

This internal structure of objects is twofold: the one face that is invariant, the other one that is ever-changing. Symmetry plays a key role in tracing the metamorphosis of form through space and through time. Lattice theory provides the means to partially order decompositions of shape, with shape rule application. Symmetry is also a helper for a conscious organization of space, from the symmetry group point of view one can enumerate all arrangements and choose the best one. A sequential analysis of the three floor plans of the Smith house will reveal how various sub-symmetries of the dihedral D_2 are systematically superimposed. As a whole, each floor plan - xy plane - does possess the full symmetry of the rectangle; even some sub-shapes conform to some subsymmetries. By extracting sub-shapes, which maximize the representation of some particular subsymmetries of the rectangle, one can construct various diagrams to illustrate the overlay of symmetries involved in the floor at each level. Thus, the final design displays an abundance of symmetries within the parts while negating the strict symmetry of the whole, and can equally assemble the lattice by type (V - S - H) to get the whole. Sub-Symmetries may be like maps; if some insights are difficult to decoding by using the map of architecture itself: orthographic projection [plan - section - elevation], the partial order lattice may be a cue, a device, an instrument for unlocking architecture from its own representation.

6.4. Discussion

A brief overview of the work has been given and some of its limitations and promises have been briefly discussed. An array of future projects based on the work presented so far was briefly mentioned as well including advances so much in the model itself as in the wider epistemological directions it points to and belongs in. Future work in the design of the model includes exploration on a) the dimensionality of the representations and the symmetry groups that are taken into account, b) the topology of the underlying grid and the regulating lines of the designs that are analyzed in this model; c) the affinity with a shape grammar formalism and particularly the design of new languages of designs that are based on the findings of the model; and d) an automation of the whole process to give automated reports with the parts of the model that exhibit specific symmetries. Other future work seeks a more profound look on the findings of the model and a critical assessment of the ways these findings support or not existing discourses, help construct new discourses, and genuinely contribute to issues pertaining to complexity and emergence in formal composition in architectural design.

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